AN INDEX FORMULA FOR PERTURBED DIRAC OPERATORS ON LIE MANIFOLDS

CATARINA CARVALHO AND VICTOR NISTOR

Abstract. We give an index formula for a class of Dirac operators coupled with unbounded potentials. More precisely, we study operators of the form $P := \not{D} + V$, where $\not{D}$ is a Dirac operators and V is an unbounded potential at infinity on a possibly non-compact manifold M_0. We assume that M_0 is a Lie manifold with compactification denoted M. Examples of Lie manifolds are provided by asymptotically Euclidean or asymptotically hyperbolic spaces. The potential V is required to be such that V is invertible outside a compact set K and V^{-1} extends to a smooth function on $M \smallsetminus K$ that vanishes on all faces of M in a controlled way. Using tools from analysis on non-compact Riemannian manifolds, we show that the computation of the index of P reduces to the computation of the index of an elliptic pseudodifferential operator of order zero on M_0 that is a multiplication operator at infinity. The index formula for P can then be obtained from the results of [17]. The proof also yields similar index formulas for Dirac operators coupled with bounded potentials that are invertible at infinity on asymptotically commutative Lie manifolds, a class of manifolds that includes the scattering and double-edge calculi.

Contents

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Introduction

Perturbed Dirac operators $\not{D} + V$ and operators $\Delta + V$ of Schrödinger type on non-compact manifolds play an important role in Quantum Mechanics, Conformal Field Theory, and in other areas. Partly for this reason, the index theory for this kind of operators has been the subject of extensive research [5, 6, 12, 13, 14, 16, 23, 22, 21, 30, 42].

The purpose of this paper is to give an index formula for Dirac operators operators coupled with unbounded potentials on even-dimensional Lie manifolds, a class of non-compact manifolds M_0 whose structure at infinity is controlled by a Lie algebra of vector fields tangent to the boundary of a suitable (given) compactification M. We also find an index formula for operators coupled with bounded potentials on a subclass of Lie manifolds that are commutative at infinity (see Definition 2.6).

Lie manifolds, or manifolds with a Lie structure at infinity, were introduced and studied in [3]. There is a natural algebra of differential operators associated to any such manifold that contains all the classical geometric operators, such as the the Dirac operator [4]. One also defines a suitable algebra of pseudodifferential operators on any Lie manifold [1], which happens to be related to an algebra of pseudodifferential operators on a differentiable groupoid. For many of these algebras Ψ^* of pseudodifferential operators on manifolds with corners, the Fredholmness of $P \in M_n(\Psi^*)$ can be characterised by the invertibility of a symbol class that consists of two components: the principal symbol $\sigma_0(P)$ and a symbol at the boundary $\sigma_\partial(P)$, also called the *indicial operator* associated to P. Thus a pseudodifferential operator compatible with the Lie manifold structure is Fredholm if, and only if, the following two conditions are satisfied: the usual ellipticity and the invertibility in the so-called indicial algebra at the boundary. The Fredholm conditions relevant for our case are discussed in Propositions 2.4 and 2.8.

Let M_0 be an even-dimensional Riemannian Lie manifold, with compactification to a manifold with corners M and $\mathcal{V}$ be the Lie algebra of vector fields tangent to the faces of M and defining the structure at infinity of M_0 (for precise definitions see §.2.1).

Let W be a Clifford module over M endowed with an admissible connection and let $\not{D} : \mathcal{C}^\infty(M; W) \to \mathcal{C}^\infty(M; W)$ be the associated generalized Dirac operator. Let us denote by $\{x_k\}$ the boundary defining functions of the hyperfaces of M. We shall consider operators of the form

$$T' = \not{D} + V := \not{D}\widehat{\otimes}1 + 1\widehat{\otimes}V : \mathcal{C}_c^\infty(M_0; W \otimes E) \to \mathcal{C}_c^\infty(M_0; W \otimes E), \tag{1}$$

where the potential $V \in \mathrm{End}(E)$ is of the form $V = f^{-1}V_0$ with

$$f := \Pi x_k^{a_k}, \quad a_k \in \mathbb{Z},\ a_k > 0, \tag{2}$$

and V_0 smooth on M and invertible at infinity (that is, on ∂M). We prove that T' is essentially self-adjoint acting on $L^2(M_0; W \otimes E)$. We shall denote by T the closure of T', which is hence a self-adjoint operator (odd with respect to the natural spinor grading). Let $\mathcal{D}(T)$ denote the domain of T and $\mathcal{D}(T) = \mathcal{D}(T)_+ \oplus \mathcal{D}(T)_-$ be its grading. We shall still write $T = \not{D} + V$, for simplicity. Let $W\widehat{\otimes}E$ be the tensor product $W \otimes E$ endowed with the usual grading.

Our main result, Theorem 3.13, is an index formula for the chiral operator

$$T_+ : \mathcal{D}(T)_+ \to L^2(M_0; W\widehat{\otimes}E)_-$$

similar to the usual Atiyah-Singer index formula. The proof of this theorem is obtained from a sequence of reductions, ultimately reducing our main result to the Atiyah-Singer type theorem for operators that are asymptotically multiplication at infinity [17]. Let us mention that our Theorem 3.13 is about as general as one can hope for such that a classical index formula would still apply. For instance, if one replaces V with a *bounded* potential V_0, then one expects an index formula for $\not{D} + V_0$ to involve non-local invariants similar to the eta invariant [8]. These non-local invariants would be associated to the faces at infinity. Thus, if one wants to avoid non-local invariants and have an index formula on an arbitrary Lie manifold just in terms of classical Chern characters, then one needs to require V to be unbounded at infinity. (Note, however, that on asymptotically commutative Lie manifolds, Definition 2.6, we do allow bounded potentials, and the calculation in this case is an important ingredient in the proof; see below for more details.) Moreover, imposing some structure at infinity also seems to be necessary and is usually done in practice. This justifies why we consider Lie manifolds and not more general non-compact manifolds. See [14] and [30] for some related approaches.

Most of the known results on the index of perturbed Dirac operators on non-compact manifolds cited above make use of crucial properties of Dirac operators, namely relative index theorems, trace formulas, or boundary conditions. In this paper, our index formula for $\not{D} + V$, with V bounded, is obtained from a general index theorem for a suitable class of pseudodifferential operators and in fact most of our results hold in the setting of pseudodifferential operators. For bounded potentials, however, we need to assume that our Lie manifold is asymptotically commutative (or commutative at infinity), Definition

2.6. A similar approach, in the bounded potentials case, is contained in [30], for odd-dimensional manifolds, where Melrose's index formula for (families of) scattering operators is used to derive an index theorem for perturbed pseudodifferential operators, so-called Callias-type operators, with bounded potentials. It is shown there that the index can be computed from invariants at the corner $S^*_{\partial M}M$. Note that the scattering structure is just a particular case of the asymptotically commutative Lie structures we consider here. (It is easily seen that Theorem 1.5 extends to the case of families, so our results can also be formulated in this setting.) To get the result for unbounded potentials, we need harder results from analysis, so we stick to differential operators, but this results holds for arbitrary Lie manifolds.

Let us now review the sequence of reductions that lead to Theorem 3.13. At the same time, we will review the contents of the paper (but in the inverse order of the sections). The first step is to write

$$T = \not{D} + V = f^{-1/2} Q f^{-1/2}, \qquad \text{with } Q := f^{1/2} \not{D} f^{1/2} + V_0,$$

which we show in Section 3 to have the same index as T. We then consider a new Lie manifold structure $(M, \mathcal{W})$ on M_0 using

$$\mathcal{W} = f\mathcal{V} := (\Pi x_k^{a_k})\mathcal{V}, \tag{3}$$

It turns out that $Q \in \mathrm{Diff}_{\mathcal{W}}(M)$, which justifies the introduction of the new Lie manifold structure $(M, \mathcal{W})$. Moreover, Q itself is a Dirac operator coupled with the *bounded* potential V_0. What makes the index of such an operator computable is the fact that the structural Lie algebra of vector fields $\mathcal{W}$ defining Q is commutative at infinity, or, to put this in another way, the indicial algebra of $\mathcal{W}$ is commutative. A Lie manifold with this property will be called *asymptotically commutative*. The analysis on the new Lie $(M, \mathcal{W})$ manifold turns out to be much easier. The index of the operator $Q = \not{D} + V_0$ associated to (general) asymptotically commutative Lie manifolds with V_0 bounded, but invertible at infinity, is obtained in Theorem 3.7 using results of Section 2.

The analytical properties of general, not necessarily even-dimensional, asymptotically commutative Lie manifolds $(M, \mathcal{W})$ and the index of operators on these spaces are studied in Section 2. We show that fully elliptic operators in $\Psi^*(M; \mathcal{W})$ can be deformed continuously to operators in $\Psi^*(M; \mathcal{W})$ that are *asymptotically multiplication* on M_0. We thus obtain an index theorem for fully elliptic pseudodifferential operators on general asymptotically commutative Lie manifolds (that is, of the form $(M, \mathcal{W})$ with $\mathcal{W}$ commutative at infinity), Theorem 2.9, generalizing known results for the scattering and double edge operators [33, 36]. Let us also mention that the case of asymptotically commutative Lie manifolds includes the important case of asymptotically Euclidean manifolds. The index formula for fully elliptic operators on asymptotically commutative Lie manifolds follows then from the results of [17]. This reduction is achieved in Section 1. We remark that all the results in Sections 1 and 2 do not assume that our manifolds are even-dimensional.

It is a classical result that on a compact manifold M_1, a pseudodifferential operator P of order m defines a Fredholm operator $H^s(M_1) \to H^{s-m}(M_1)$ if, and only if, it is elliptic. In other words, on a compact manifold, ellipticity is equivalent to Fredolmness. By contrast, on non-compact manifolds, ellipticity is typically only a necessary, but not sufficient condition to ensure Fredholmness; stronger conditions on an operator P are required to obtain that P is Fredholm. For example, on an asymptotically commutative Lie manifold, the Fredholm condition is still controlled by the invertibility of a function, which this time is an extension of the principal symbol, and hence is defined on an extension of the cosphere bundle. This phenomenon is studied in Section 1, where an index theorem is proved for such operators by reducing to the case of operators that are multiplication at infinity (which was studied in [17]). In particular, we obtain in that section an index theorem for asymptotically multiplication operators.

We shall assume throughout most of this paper that M_0 is a non-compact Lie manifold with compactification M, althought some of our results of the earlier sections may be true for more general non-compact manifolds. For instance, the index theorem of [17] is valid without any assumption on M_0.

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1. Asymptotically multiplication operators

In this section, we review some basic concepts and results to be used in what follows, leading to the Atiyah-Singer index theorem in the setting of *non-compact* manifolds and operators that are multiplication outside a compact set (or asymptotically so). Here, we keep the manifolds quite general, while we consider a class of operators that inherits naturally the properties of the compact manifold case. For simplicity, we assume that M_0 is endowed with a metric g and that, as a topological space, it is the interior of a compact manifolds with corners M such that TM restricts to TM_0 on M_0. We let n be the dimension on M_0, which in this and the following section may be arbitrary, but in the last section will be assumed to be even.

1.1. **General calculus.** We consider for now a smooth manifold M_0 without boundary, not necessarily compact, and a smooth vector bundle E over M_0 that is trivial outside a compact set in M_0. We denote by $d\,\mathrm{vol}_g$ the volume form on M_0 defined by the metric. We also assume that E is endowed with a Hermitian metric, which is the trivial (product) metric close to infinity. Typically, M_0 will coincide with the interior of a given compact manifold with corners M.

We first make a short review of main results of the theory of pseudodifferential operators on M_0 that we need in this paper, in the setting of operators that are multiplication by a smooth function outside a compact set. We follow closely the approach of [17] (see however also [28] or [44] for general references).

Let us recall that a smooth function $p : W \times \mathbb{R}^n \longrightarrow \mathbb{C}^{N\times N}$ defines a symbol in the class $S^m(W \times \mathbb{R}^n)$ of *symbols of order* m if, and only if, for any compact set $K \subset W$ and multi-indices α, β, there exists $\mathcal{C}_{K,\alpha,\beta} > 0$ such that $|\partial_x^\alpha \partial_\xi^\beta p(x,\xi)| \leq \mathcal{C}_{K,\alpha,\beta}(1+|\xi|)^{m-|\beta|}$, for all $x \in K$ and $\xi \in \mathbb{R}^n$. An operator $P : \mathcal{C}_c^\infty(M_0; E) \to \mathcal{C}^\infty(M_0; E)$ is said to be in the class $\Psi^m(M_0; E)$ of *pseudodifferential operators of order* m on M_0 if, for any coordinate chart W of M_0 trivializing E and for any $h \in \mathcal{C}_c^\infty(W)$, $hPh : \mathcal{C}_c^\infty(W)^N \to \mathcal{C}_c^\infty(W)^N$ is a matrix of pseudodifferential operators of order m on W, that is, $hP(hu) = p(x, D)u$, with

$$hP(hu)(x) = p(x, D)u(x) := (2\pi)^{-n} \int_{\mathbb{R}^n} p(x,\xi)\widehat{u}(\xi)e^{ix\cdot\xi}d\xi, \tag{4}$$

where $\widehat{u}$ denotes the Fourier transform of u and $p \in S^m(W \times \mathbb{R}^n)$.

We shall work only with classical symbols, that is, symbols that have asymptotic expansions $p \sim \sum p_{m-k}$ with $p_{m-k} \in S^{m-k}(W\times\mathbb{R}^n)$ positively homogeneous of degree $m-k$ in ξ. Let us denote by $\pi : T^*M_0 \to M_0$ the cotangent bundle of M_0. The leading term p_m in the expansion of $p(x,\xi)$ as a classical symbol defines a smooth section of the bundle $\mathrm{End}(E)$ over the cotangent bundle T^*M_0, the *principal symbol* of P, which is a smooth bundle homomorphism $\sigma_m(P) : \pi^*E \to \pi^*E$, positively homogeneous on the fibers of T^*M_0. By choosing a metric on TM_0, the class of principal symbols can be identified with $\mathcal{C}^\infty(S^*M_0; \mathrm{End}(E))$ where S^*M_0 is the unit sphere bundle of the cotangent bundle. An operator P is said to be *elliptic* if $\sigma_m(P)$ is invertible on S^*M_0. We shall regard the cosphere bundle S^*M_0 as the boundary of T^*M_0 using a radial compactification of each fiber. It is in this sense that we shall often extend the principal symbol of an order zero pseudodifferential operator to T^*M_0.

Under certain assumptions that will be satisfied in our setting, we have that if $P \in \Psi^m(M_0; E)$, then $P_0 := P(1 + P^*P)^{-1/2} \in \Psi^0(M_0; E)$, with P^* the formal adjoint, and P is Fredholm, respectively, elliptic, if, and only if, P_0 is, with $\mathrm{ind}(P) = \mathrm{ind}(P_0)$. Moreover, $\sigma_m(P)$ is homotopic to $\sigma_0(P_0)$, as sections of S^*M_0 (the unit sphere bundle of T^*M_0). Hence, for the purposes of index theory, we will mainly be concerned with operators of order 0.

We start with considering the class of pseudodifferential operators that are *multiplication outside a compact*, defined as

$$\Psi^0_{mult}(M_0; E) := \{P = P_1 + p,\ P_1 \in \Psi^0(M_0; E) \text{ has a compactly supported distribution kernel and } p \in \mathrm{End}(E) \text{ is bounded}\}. \tag{5}$$

(For $m < 0$, we consider $\Psi^m_{mult}(M_0; E) := \Psi^0_{mult}(M_0; E) \cap \Psi^m(M_0; E)$). We have that any operator in $\Psi^0_{mult}(M_0; E)$ is properly supported and that $\Psi^0_{mult}(M_0; E)$ is a $*$-algebra (see [17] for details). Moreover, denoting by $S^0_{mult}(T^*M_0; E)$ the set of bounded symbols in

$S^0(T^*M_0; E)$ that are constant on the fibres of $T^*M_0 \to M_0$ outside a compact of M_0, we have that there is a well-defined symbol map

$$\sigma_0 : \Psi^0_{mult}(M_0; E) \to S^0_{mult}(T^*M_0; E)/S^{-1}_{mult}(T^*M_0; E), \tag{6}$$

which is a surjective $*$–homomorphism with $\ker(\sigma_0) = \Psi^{-1}_{mult}(M_0; E)$, as it is the case if M_0 is compact. Moreover, we have that $P \in \Psi^0_{mult}(M_0; E)$ is always bounded as an operator on $L^2(M_0; E)$, and that $\Psi^{-1}_{mult}(M_0; E)$ consists of compact operators, again as in the classical case of compact manifolds.

We endow the class of symbols $S^0_{mult}(T^*M_0; E)$ with the sup-norm, as a section of $\mathrm{End}(E)$ over T^*M_0. Note that $S^0_{mult}(T^*M_0; E)$ can be identified with the class of bounded sections in $\mathcal{C}^\infty(S^*M_0; \mathrm{End}(E))$ that are constant on the fibers of S^*M_0 outside a compact $K \subset M_0$, and this is consistent with regarding S^*M_0 as the boundary of (the radial fibrewise compactification of) T^*M_0. (Recall that E is trivialized outside a compact set, so "constant in a neighborhood of infinity" does indeed make sense.)

The class $\mathcal{C}_a(S^*M_0; E)$ of *asymptotically multiplication symbols* is defined as those functions $p = p(x, \xi) \in \mathcal{C}(S^*M_0; \mathrm{End}(E))$ such that $p(x, \xi)$ is bounded in the sup-norm and, for all $\epsilon > 0$, there is a compact $K_\epsilon \subset M_0$ such that, for all $x \notin K_\epsilon$,

$$\sup_{\xi_1, \xi_2 \in S^*M_0} \|p(x, \xi_1) - p(x, \xi_2)\|_{\mathrm{End}(E_x)} < \epsilon. \tag{7}$$

Roughly speaking, the elements of $\mathcal{C}_a(S^*M_0; E)$ are continuous sections of $\mathrm{End}(E)$ over S^*M_0 that are bounded and asymptotically independent of ξ on the fibers of S^*M_0. It is easily checked that it is a C^*-subalgebra of $\mathcal{C}_b(S^*M_0; \mathrm{End}(E))$, the class of continuous, bounded sections of $\mathrm{End}(E)$.

We now define the class of *asymptotically multiplication pseudodifferential operators* as

$$\Psi^0_a(M_0; E) := \overline{\Psi^0_{mult}(M_0; E)} \subset \mathcal{B}(L^2(M_0; E)), \tag{8}$$

that is, the closure of $\Psi^0_a(M_0; E)$ in the topology of bounded operators on $L^2(M_0; E)$.

The point of the following lemma is that, once we consider completions, we will need to replace operators that are multiplication at infinity with asymptotically multiplication operators.

Lemma 1.1. *The principal symbol defines a natural map $\Psi^0_{mult}(M_0; E) \to \mathcal{C}_a(S^*M_0; E)$, which extends by continuity to a surjective map $\Psi^0_a(M_0; E) \to \mathcal{C}_a(S^*M_0; E)$.*

Proof. We show that $\mathcal{C}_a(S^*M_0; E)$ coincides with the closure of

$$\begin{aligned} \Xi := \{\, p \in \mathcal{C}^\infty_b(S^*M_0; \mathrm{End}(E)),\ \text{there exists } K \subset M_0 \text{ compact} \\ \text{such that } p(x, \xi) \text{ is independent of } \xi \text{ if } x \notin K \,\}, \end{aligned} \tag{9}$$

and the result then follows as in the compact case. It is readily checked that any p in the closure of (9) is asymptotically multiplication. For the converse, let $p \in \mathcal{C}_a(S^*M_0; \mathrm{End}(E))$

and take $\widetilde{p} \in \mathcal{C}_b^\infty(S^*M_0; \mathrm{End}(E))$ such that $\|\widetilde{p} - p\|_{\sup} < \epsilon$. Let $K \subset M_0$ be compact such that $\|p(x,\xi_1) - p(x,\xi_2)\|_{\mathrm{End}(E_x)} < \epsilon$, for all $x \notin K$, $\xi_1, \xi_2 \in S^*M_0$, and let $\phi \in \mathcal{C}_c^\infty(S^*M_0)$ be such that $\operatorname{supp}\phi \subset M_0 - K$, $0 \le \phi \le 1$, and $\phi(x,\xi) = 1$, for $x \notin K'$, with K' compact such that $K \subset int(K')$. Define

$$q(x,\xi) := (1 - \phi(x,\xi))\, \widetilde{p}(x,\xi) + \phi(x,\xi)\, \widetilde{p}(s(x)),$$

where s is a fixed smooth section of S^*M_0 (which exists since every connected non-compact manifold has a nowhere vanishing vector field). Then $q(x,\xi) \in \mathcal{C}_b^\infty(S^*M_0; \mathrm{End}(E))$ and for $x \notin K'$, $q(x,\xi) = \widetilde{p}(s(x))$ is independent of ξ. Moreover,

$$\begin{aligned}\|q - p\|_{\sup} \le \sup_{\substack{x\in K' \\ \xi \in S^*M_0}} \|\widetilde{p}(x,\xi) - p(x,\xi)\|_{\mathrm{End}(E_x)} + \sup_{\substack{x\notin K, \\ \xi \in S^*M_0}} \|\widetilde{p}(s(x)) - p(x,\xi)\|_{\mathrm{End}(E_x)} \\ \le 2\|\widetilde{p} - p\|_{\sup} + \sup_{\substack{x\notin K, \\ \xi \in S^*M_0}} \|p(s(x)) - p(x,\xi)\|_{\mathrm{End}(E_x)} \le 3\epsilon.\end{aligned}$$

Hence, p lies in the closure of Ξ defined in Equation (9), and that concludes our proof. □

The following result can be proved much as in the compact case.

Proposition 1.2. *The principal symbol map (6) is continuous and the following sequence of C^*-algebras is exact*

$$0 \longrightarrow \mathcal{K}(M_0; E) \longrightarrow \Psi_a^0(M_0; E) \xrightarrow{\sigma_0} \mathcal{C}_a(S^*M_0; E) \longrightarrow 0. \tag{10}$$

where now $\sigma_0 : \Psi_a^0(M_0;E) \to \mathcal{C}_a(S^*M_0;E)$ denotes the extension by continuity of the classical principal symbol map $\sigma_0 : \Psi_{mult}^0(M_0;E) \to \mathcal{C}_a(S^*M_0;E)$.

Proof. The exactness at $\mathcal{C}_a(S^*M_0;E)$ follows from Lemma 1.1. Using a partition of unity and the fact that our result is true in the compact case, we see that $\Psi_{mult}^{-1}(M_0;E) \subset \mathcal{K}$ as a dense subset. This proves the exactness at $\mathcal{K}(M_0;E)$ and the fact that $\mathcal{K}(M_0;E)$ is contained in the kernel of σ_0.

As in the classical case of compact manifolds, the difficult case is to prove that if an operator $T \in \Psi_a^0(M_0;E)$ is in the kernel of σ_0, then it is compact. Let then

$$T_n \in \Psi_{mult}^0(M_0;E), \quad T_n \to T \quad \text{and} \quad \sigma_0(T_n) \to 0.$$

Then we can replace the sequence T_n with a sequence of operators that are *zero* in a neighborhood of infinity. Also, let $\psi \in \mathcal{C}_c^\infty(M_0)$ have the support in a local coordinate chart. Then $\psi T_n \psi \to \psi T \psi$ and $\sigma_0(\psi T_n \psi) \to 0$. Using the case of a compact manifold, we see that $\psi T \psi$ is a compact operator. From this we infer that $\psi_1 T \psi_2$ is also compact for *any* compactly functions ψ_1 and ψ_2. (One way to prove this is to consider first the case when ψ_1 and ψ_2 have disjoint supports). Let $0 \le \ldots \le \psi_k \le \psi_{k+1} \le \ldots \le 1$ be an increasing sequence of compactly functions such that $\psi_n(x) \to 1$ for all x. (We are assuming here that M_0 is σ-compact, which is always the case if M_0 has a compactification.)

We claim that $\psi_k T \psi_k \to T$. Since $\psi_k T \psi_k$ is compact for any k, it will follow that T is also compact. To prove our claim, let $\epsilon > 0$ and choose n such that $\|T - T_n\| < \epsilon/3$. Then we can find k_0 such that $\|\psi_k T_n \psi_k - T_n\| \leq \epsilon/3$ for $k \geq k_0$ since T_n is assumed to be zero outside a compact set. Then

$$\|T - \psi_k T \psi_k\| \leq \|T - T_n\| + \|T_n - \psi_k T_n \psi_k\| + \|\psi_k (T_n - T)\psi_k\| \leq \epsilon$$

for $k \geq k_0$. This completes our proof. □

It follows from Proposition 1.2 that $P \in \Psi^0_a(M_0; E)$ is a Fredholm operator if, and only if, its full symbol is invertible in $\mathcal{C}_a(S^*M_0; E)$ or, equivalently, in $\mathcal{C}_b(S^*M_0; E)$. See also [24] for a discussion of Fredholm operators on non-compact manifolds.

1.2. **The Atiyah-Singer index theorem.** We now review the Atiyah-Singer index formula, applied to asymptotically multiplication operators. (See for instance [9, 35] for the details on the constructions below).

First, we define operators acting between sections of two different vector bundles. Let E, F be vector bundles over M_0, with $E \cong F$ outside a compact. We define $\Psi^0_{mult}(M_0; E, F)$ as the subclass of $\Psi^0_{mult}(M_0; E \oplus F)$ of those operators that induce $P : \mathcal{C}^\infty_c(M_0; E) \to \mathcal{C}^\infty(M_0; F)$. We have so also $S^0_{mult}(T^*M_0; E, F) \subset S^0_{mult}(T^*M_0; E \oplus F)$ and all the results above hold, except that if $P \in \Psi^0_{mult}(M_0; E, F)$ then $P^* \in \Psi^0_{mult}(M_0; F, E)$, so we leave the setting of C^*-algebras. In any case, an analogue of the exact sequence given in Proposition 1.2 holds.

We now associate a K-theory class to an elliptic Fredholm operator in $\Psi^0_a(M_0; E, F)$.

Lemma 1.3. *For any elliptic, bounded $Q \in \Psi^0_{mult}(M_0; E, F)$ such that $q := \sigma_0(Q)$, here is a natural class $[\sigma_0(Q)] := [\pi^*E, \pi^*F, q]$ in the compactly supported K-theory of T^*M_0 obtained by extending q to an invertible map outside a compact set of TM_0 that is constant along the fibers of $TM_0 \to M_0$ outside a compact set. This K-theory class is such that the Fredholm index of Q depends only on $[\sigma_0(Q)]$.*

Proof. In order to associate a K-theory class to a Fredholm operator in $\Psi^0_a(M_0; E, F)$, we start with noting that if $Q \in \Psi^0_{mult}(M_0; E, F)$ is such that $q := \sigma_0(Q)$ is invertible, then q defines an isomorphism outside a compact subset of T^*M_0 by homogeneity and the fact that it is constant on the fibres outside a compact in M_0. Hence, regarding $q(x, \xi)$ as a bundle map $\pi^*E \to \pi^*F$, $\pi : T^*M_0 \to M_0$, we obtain the desired definiton of $[\sigma_0(Q)] := [\pi^*E, \pi^*F, q]$ as in [7, 29]. The dependence of the index only on $[\sigma_0(Q)]$ follows as in the classical case by noticing that Q is Fredholm as long as the principal symbol is invertible as in [17]. □

Given now a Fredholm operator $P \in \Psi^0_a(M_0; E, F)$ with invertible symbol $\sigma_0(P) \in \mathcal{C}_a(S^*M_0; E, F)$, by Proposition 1.2, we can take $q \in S^0_{mult}(T^*M_0; E, F)$ sufficiently close to $\sigma_0(P)$ such that $t\sigma_0(P)+(1-t)q$, $t \in [0,1]$, is an homotopy through invertible symbols. We define the symbol class of P as

$$[\sigma_0(P)] := [\pi^* E, \pi^* F, q] \in K^0(T^* M_0). \tag{11}$$

This class is independent of q. If we take $q = \sigma_0(Q)$, with $Q \in \Psi^0_{mult}(M_0; E, F)$, we have $\operatorname{ind}(P) = \operatorname{ind}(Q)$. Moreover, if two Fredholm operators have the same symbol class, then their indices coincide, and, in fact there is a well-defined (analytic) index map

$$\operatorname{ind} : K^0(TM_0) \to \mathbb{Z}, \quad [\sigma_0(P)] \mapsto \operatorname{ind}(P), \tag{12}$$

where $P \in \Psi^0_a(M_0; E, F)$ and we use the metric to identify canonically T^*M_0 with TM_0. We summarize the above discussion in the following lemma extending Lemma 1.3.

Lemma 1.4. *For any elliptic, bounded $Q \in \Psi^0_a(M_0; E, F)$ such that $q := \sigma_0(Q)$, there is a natural class $[\sigma_0(Q)] := [\pi^* E, \pi^* F, q]$ in the compactly supported K-theory of T^*M_0 obtained by extending q to an invertible map outside a compact set of TM_0 that is asymptotically constant along the fibers of $TM_0 \to M_0$. This K-theory class is such that the Fredholm index of Q depends only on $[\sigma_0(Q)]$.*

For a manifold X, we let $H^*(X)$, respectively $H^*_c(X)$, denote the cohomology, respectively the compactly supported cohomology, of X. Recall that throughout this section, we assume that M_0 is the interior of a compact manifold with corners M. Let $\overline{TM}$ be the radial compactification of the tangent bundle to M. Then the pair $(\overline{TM}, \partial\overline{TM})$ is homeomorphic to the similar pair associated to a manifold with boundary. Hence the (even) Chern character yields a map

$$\mathrm{ch}_0 : K^0(TM_0) \to H^{2*}_c(TM_0) = H^{2*}(\overline{TM}, \partial\overline{TM}).$$

(We will also consider later on the odd Chern character ch_1 defined on K^1.) Let also $Td(T_{\mathbb{C}}M) \in H^*(M)$ denote the Todd class of the complexified tangent bundle $TM \otimes \mathbb{C}$. Note that since TM is oriented, there is a well-defined fundamental class $[TM_0] \in H_{2n}(\overline{TM}, \partial\overline{TM})$ (see for instance [35] for details on these constructions).

The following result is an immediate extension of a result in [17] from operators that are multiplication outside a compact to operators that are only asymptotically so. Let $\pi : \overline{TM} \to M$ denote the natural projection. We have $ch_0[\sigma_0(P)] \in H^{2*}(\overline{TM}, \partial\overline{TM})$ and $\pi^* Td(T_{\mathbb{C}}M) \in H^{2*}(\overline{TM})$ so their product is in $H^{2*}(\overline{TM}, \partial\overline{TM}) = H^{2*}_c(TM_0)$.

Theorem 1.5. *Let $P \in \Psi^0_a(M_0; E, F)$ be such that $\sigma_0(P)$ is invertible in $\mathcal{C}_a(S^*M_0; E, F)$. Then P is Fredholm and*

$$\operatorname{ind}(P) = (-1)^n \, \mathrm{ch}_0[\sigma_0(P)] \pi^* Td(T_{\mathbb{C}}M)[TM_0],$$

where $[\sigma_0(P)]$ is defined using Lemma 1.4.

Proof. The fact that P is Fredholm follows from Proposition 1.2. The rest of the proof follows from the discussion before the statement of this theorem. Indeed, let $P \in \Psi^0_a(M_0; E, F)$ be elliptic. Then we can find $P_0 \in \Psi^0_{mult}(M_0; E, F)$ that is close enough to P such that the straight line joining P and P_0 consists of invertible operators. Both the left hand side and the right hand side of our index formula are homotopy invariant. For P_0 they are equal by [17]. For P, they will be therefore equal as well, by homotopy invariance. □

1.3. **Comparison spaces.** In this subsection, we extend by deformation the index formula of Theorem 1.5 to certain pseudodifferential operators on noncompact manifolds that extend to the compactification M of M_0 in a suitable sense. More precisely, we require the principal symbols of our operators to extend to a so-called "comparison space" and there is an invertible complete symbol at the boundary. We thus generalize the approach in [23, 36], using homotopy to asymptotically multiplication symbols.

Recall that M is a compactification of M_0 to a manifold with corners. In this section, we fix a vector bundle A over M such that $A|_{M_0} \cong TM_0$ (later, when we consider Lie structures, such an A will be naturally associated to M_0.) Denote by $\overline{A}$ the fiber-wise radial compactification of A, so $\overline{A}$ is a manifold with corners that fibers over M with fibers closed balls of dimension n. We identify A with A^* using a fixed metric. Let $(S^*A)_{\partial M}$ be the restriction of the cosphere bundle S^*A to the boundary ∂M of M. Define

$$\Omega := \partial(\overline{A}) = (S^*A) \cup \overline{A}|_{\partial M} \tag{13}$$

such that

$$\mathcal{C}(\Omega) = \{(f, g) \in \mathcal{C}(S^*A) \oplus \mathcal{C}(\overline{A}|_{\partial M}) : f|_{(S^*A)_{\partial M}} = g_{(S^*A)|_{\partial M}}\}.$$

The space Ω will play an important role in what follows. It is closely related to a similar space introduced by Cordes and his colaborators in his work on Gelfand theory for noncompact manifolds [19, 18].

Let $\Psi_A(M_0; E) \subset \Psi^0(M_0; E)$ be a $*$-algebra of order 0, bounded, pseudodifferential operators. We say that Ω is a *comparison space* for $\Psi_A(M_0; E)$ if there is a surjective homomorphism

$$\sigma_{full} : \overline{\Psi_A(M_0; E)} \to \mathcal{C}(\Omega) \tag{14}$$

such that $\sigma_{full}(P)|_{S^*M_0} = \sigma_0(P)$, with kernel included in the algebra of compact operators. We call σ_{full} a *full* symbol and write $\sigma_{full} = (\sigma_0, \sigma_\partial)$, where

$$\sigma_\partial : \overline{\Psi_A(M_0; E)} \to \mathcal{C}(\overline{A}|_{\partial M})$$

is the *boundary symbol morphism.* An operator with invertible full symbol is called *fully elliptic.* We shall give an index formula for fully elliptic operators in this setting, reducing to asymptotically multiplication operators.

We see first that any function in $\mathcal{C}(\Omega)$ can be homotoped over the interior to an asymptotically multiplication symbol. Since the fibres of $\overline{A}$ are isomorphic to the n-dimensional half sphere $\mathbb{S}^n_+$ and hence contractible, we have that Ω is homotopy equivalent to the space $\widetilde{\Omega}$ obtained from the cosphere bundle S^*A by collapsing the fibers above points of the boundary. More precisely, $\widetilde{\Omega} := (S^*A)/\sim$, with $(x,\xi) \sim (x,\xi')$, for $x \in \partial M$, $\xi, \xi' \in S^*A_x$. Since

$$\mathcal{C}(\widetilde{\Omega}) \cong \{f \in \mathcal{C}(\Omega) : f \text{ constant on fibers over } \partial M\} \\ \cong \{f \in \mathcal{C}(S^*A) : f \text{ constant on fibers over } \partial M\},$$

we conclude that every $f \in \mathcal{C}(\Omega)$ is canonically homotopic to some $\widetilde{f} \in \mathcal{C}(\Omega)$ constant on the fibres of $\overline{A}|_{\partial M} \to \partial M$ (this is achieved by a homotopy equivalence between Ω and $\widetilde{\Omega}$). Moreover, if f is invertible, the canonical homotopy between f and $\widetilde{f}$ is through invertible functions. We now have the following:

Lemma 1.6. *If $f \in \mathcal{C}(\Omega, E)$ is constant on fibers of $\overline{A}|_{\partial M} \to \partial M$, then $f_0 := f|_{S^*M_0} \in \mathcal{C}_a(S^*M_0, E)$. In particular, let $f \in \mathcal{C}(\Omega, E)$, then f is homotopic to $\widetilde{f} \in \mathcal{C}(\widetilde{\Omega}, E) \subset \mathcal{C}(\Omega, E)$ and hence it satisfies $f_0 := \widetilde{f}|_{S^*M_0} \in \mathcal{C}_a(S^*M_0, E)$.*

Proof. We consider only the scalar case. Let $p \in \mathcal{C}(\partial M)$ be such that $f_{(S^*A)_{\partial M}} = p$. It suffices to show that, given $\epsilon > 0$, there is a neighborhood U of ∂M in M such that for $y \in M_0 \cap U$, $\xi, \xi' \in S^*_y M_0$,

$$\|f_0(y,\xi) - f_0(y,\xi')\| < \epsilon \tag{15}$$

so that (7) follows with $K = M \backslash U$. We can do this locally, so assume U_x is a neighborhood of $x \in \partial M$ such that $\pi^{-1}(U_x) \cong U_x \times S^{n-1}$, $\pi : S^*A \to M$. Let

$$V_x := \{(y,\xi) \in U_x \times S^{n-1} \colon \|f(y,\xi) - p(x)\| < \epsilon/2\}.$$

By continuity of f in S^*A, we have that V_x is open in $U_x \times S^{n-1}$. Moreover, since $f(y,\xi) = p(y)$, $y \in \partial M$, it contains $W_x \times S^{n-1}$ for some $W_x \subset \partial M$ open. Hence, V_x can be taken as $\widetilde{U}_x \times S^{n-1}$, for some open neighborhood $\widetilde{U}_x$ of W_x in M so that (15) holds if $y \in \widetilde{U}_x$. The last statement follows from the fact that Ω and $\widetilde{\Omega}$ are homotopy equivalent. □

Let us notice that for $f \in \mathcal{C}(\widetilde{\Omega}, E)$, the restriction $f|_{S^*M_0}$ completely determines f since S^*M_0 is dense in $\widetilde{\Omega}$. Let now $P \in \overline{\Psi^0_A(M;E)}$ have invertible symbol

$$\phi = (\sigma_0(P), \sigma_\partial(P)) \in \mathcal{C}(\Omega, E).$$

(In the terminology introduced earlier, P is fully elliptic.) From the previous lemma, there is an invertible $\widetilde{\phi} \in \mathcal{C}(\widetilde{\Omega}, E) \subset \mathcal{C}(\Omega, E)$ homotopic through invertibles to ϕ, with $\widetilde{\sigma} := \widetilde{\phi}|_{S^*M_0} \in \mathcal{C}_a(S^*M_0, E)$ and $\sigma := \sigma_0(P)$ and $\widetilde{\sigma}$ are homotopic (over M_0). If we let

$\widetilde{\sigma} = \sigma_0(\widetilde{P})$, for some $\widetilde{P} \in \Psi_a^0(M_0; E) \cap \overline{\Psi_A^0(M; E)}$, then $\widetilde{P}$ is Fredholm, since $\widetilde{\sigma}$ is invertible. Moreover, from the surjectivity of the complete symbol map (14), there is a continuous family $P_t \in \overline{\Psi_A^0(M; E)}$, $t \in [0, 1]$, lifting the homotopy between ϕ and $\widetilde{\phi}$. Hence $P_0 = P$ and $P_1 = \widetilde{P}$ and $\mathrm{ind}(P) = \mathrm{ind}(\widetilde{P})$.

We conclude also that any fully elliptic operator $P \in \overline{\Psi_A^0(M; E, F)}$ has associated a well-defined K-theory class $[\widetilde{\sigma}_{full}(P)]$ extending by homotopy the definition in Equation (11) as follows. We know that P is homotopic to some $\widetilde{P} \in \Psi_a^0(M_0; E)$ through Fredholm operators in $\Psi_A^0(M_0; E)$ and $[\widetilde{\sigma}_{full}(P)] := [\sigma_0(\widetilde{P})]$, so that

$$[\widetilde{\sigma}_{full}(P)] := [\sigma_0(\widetilde{P})] = [\pi^* E, \pi^* F, \widetilde{p}] \in K^0(TM_0), \tag{16}$$

where we assume $\widetilde{p}$ is an extension of the principal symbol of $\widetilde{P}$ to a function that is multiplication at infinity and homotopic to $\sigma_0(P)$ over the interior. In particular.

$$\mathrm{ind}(P) = \mathrm{ind}(\widetilde{P}) = \mathrm{ind}([\sigma_0(\widetilde{P})]) = \mathrm{ind}([\widetilde{\sigma}_{full}(P)]).$$

Consider now, for $P \in \overline{\Psi_A^0(M; E, F)}$ fully elliptic,

$$[\sigma_{full}(P)] := [(\sigma_0(P), \sigma_\partial(P))] \in K_1(\mathcal{C}(\Omega)) \cong K^1(\Omega) \tag{17}$$

where $\sigma_\partial(P) \in \mathcal{C}(\overline{A}_{\partial M})$ and $\sigma_0(P) \in \mathcal{C}(S^* A)$ denote the boundary and principal symbol, respectively. Let us consider the connecting map $\partial : K^1(\Omega) \to K^0(TM_0)$ in the long exact sequence of the pair $(\overline{A}, \partial \overline{A}) = (\overline{A}, \Omega)$. We summarize the above discussion to the following generalization of Lemma 1.4.

Lemma 1.7. *For any fully elliptic $P \in \overline{\Psi_A^0(M; E, F)}$ there is a natural class*

$$[\widetilde{\sigma}_{full}(P)] := [\pi^* E, \pi^* F, p]$$

in the compactly supported K-theory of $T^* M_0$ obtained by extending $\sigma_{full}(P)$ to a continuous endomorphism p invertible outside a compact set of TM_0. This K-theory class is such that the Fredholm index of P depends only on $[\widetilde{\sigma}_{full}(P)]$ and

$$[\widetilde{\sigma}_{full}(P)] = \partial[(\sigma_0(P), \sigma_\partial(P))] = \partial[\sigma_{full}(P)].$$

Let us also notice that Ω is homotopically equivalent to the boundary of an oriented smooth manifold with boundary, and hence it has a well defined fundamental class $[\Omega] \in H_{2n-1}(\Omega)$. If $[\overline{A}]$ denotes the fundamental class of $\overline{A}$ in $H_{2n}(\overline{A}, \Omega)$, then $[\Omega] = \partial[\overline{A}]$. Using the compatibility of the boundary maps in K-theory and cohomology, that is, the fact that the Chern character is a natural transformation of cohomology theories (see [40] for an extension of this result to non-commutative algebras), we obtain the following result as a consequence of the Atiyah-Singer index formula extended to operators that are asymptotically multiplication operators (Theorem 1.5).

As before, let $Td(T_{\mathbb{C}} M)$ denote the Todd class of the complexified tangent bundle of M, and $\pi : \overline{T^* M} \to M$. Also, we denote by $\pi_\Omega : \Omega \to M$ the natural projection.

Theorem 1.8. *Let Ω be a comparison space for $\Psi^0_A(M;E,F)$ and $P \in \overline{\Psi^0_A(M;E,F)}$ be fully elliptic operator (that is, an elliptic operator with $\sigma_\partial(P)$ invertible in $\mathcal{C}(\overline{A}|_{\partial M})$). Then P is Fredholm and*

$$\mathrm{ind}(P) = (-1)^n \,\mathrm{ch}_0[\widetilde{\sigma}_{full}(P)]\pi^* Td(T_{\mathbb{C}}M)[TM_0] = (-1)^n \;\mathrm{ch}_1[\sigma_{full}(P)]\pi^*_\Omega Td(T_{\mathbb{C}}M)[\Omega],$$

where $[\widetilde{\sigma}_{full}(P)]$ is defined using Lemma 1.7.

Proof. Again, the proof of the first equality follows from the discussion before the statement of the theorem. Indeed, let us choose a homotopy between P and $\widetilde{P} \in \Psi^0_a(M_0;E,F)$ through Fredholm operators $\overline{\Psi^0_A(M;E,F)}$. Both the left hand side and the right hand side(s) of the index formula of this theorem are homotopy invariant. For $\widetilde{P}$ they are equal in view of Theorem 1.5, by homotopy invariance, they will be equal also for P. To prove the last equality, we just use the fact that the Chern character is compatible with the boundary maps in K-theory and cohomology. (A proof of a generalization of this result to non-commutative algebras can be found in [40].) □

2. Index formula on asymptotically commutative Lie manifolds

From now on, we endow M_0 with the structure of a Lie manifold with compactification M and structural Lie algebra of vector fields $\mathcal{V}$ (see below for the definitions). There is associated to $(M,\mathcal{V})$ a well-behaved pseudodifferential calculus and, for operators in this calculus, Fredholm criteria follow from the pseudodifferential calculus of operators on groupoids [1, 31].

We show that if we introduce the additional assumptions on the structural Lie algebra $\mathcal{V}$ that it be *asymptotically commutative*, then there will exist a (commutative) complete symbol and hence we can apply the results in the previous section. Recall that in this section n may be arbitrary (in the following section will be assumed to be even).

2.1. **Operators on Lie manifolds.** In this section, M will denote a compact manifold with corners and $M_0 = int(M)$, as before. Also, let $\mathcal{V}_M$ denote the Lie algebra of vector fields that are tangent to all faces of M. We always assume that each hyperface $H \subset M$ is an embedded submanifold of M and hence that it has a defining function x_H (recall that this means that x_H is smooth on M, $x_H \geq 0$, $H = \{x_H = 0\}$, and $dx_H \neq 0$ on H).

We recall the main definitions of [3, 1]. We say that a Lie subalgebra $\mathcal{V} \subset \mathcal{V}_M$ is a *structural Lie algebra of vector fields* if it is a Lie algebra with respect to the Lie bracket and it is also a finitely generated, projective, $\mathcal{C}^\infty(M)$-module. By the Serre-Swan theorem, we have that there exists a vector bundle A such that $\mathcal{V} \cong \Gamma(A)$. Moreover, there is a vector bundle morphism $\rho : A \to TM$, called *anchor map*, which induces the inclusion map $\rho : \mathcal{V} = \Gamma(A) \to \Gamma(TM)$. It thus follows that A with the given structure is naturally a Lie algebroid.

Definition 2.1. A *Lie manifold* M_0 is given by a pair $(M, \mathcal{V})$ where $M_0 = int(M)$ and $\mathcal{V}$ is structural Lie algebra of vector fields such that $\rho|_{M_0} : A|_{M_0} \to TM_0$ is an isomorphism.

A metric on M_0 that is obtained from a metric on A by restriction to $A|_{M_0} \cong TM_0$ will be called a *compatible metric* on M_0. Any two such metrics are Lipschitz equivalent. We fix one of these metrics on M_0 in what follows.

To a Lie manifold $(M, \mathcal{V})$ we associate the algebra $\mathrm{Diff}_{\mathcal{V}}(M_0)$ of *$\mathcal{V}$-differential operators on* M_0, defined as the enveloping algebra of $\mathcal{V}$ (generated by $\mathcal{V}$ and $\mathcal{C}^\infty(M)$). It was shown in [3] that $\mathrm{Diff}_{\mathcal{V}}(M_0)$ contains all geometric operators on M_0 associated to a compatible metric, such as the Dirac and generalized Dirac operators. (This property of $\mathrm{Diff}_{\mathcal{V}}(M_0)$ will be used in Section 3.) One defines differential operators acting between sections of vector bundles E, F over M as

$$\mathrm{Diff}_{\mathcal{V}}(M_0; E, F) := e_F M_N(\mathrm{Diff}_{\mathcal{V}}(M_0)) e_E,$$

where e_E, e_F are projections onto $E, F \subset M \times \mathbb{C}^N$. In [1], a class of pseudodifferential operators associated to a given Lie structure at infinity is defined by a process of microlocalizing $\mathrm{Diff}_{\mathcal{V}}(M_0; E, F)$. We outline this construction below.

Recall that we first define the class $S^m(A^*) \subset C^\infty(A^*)$ as functions satisfying the usual symbol estimates on coordinate patches trivializing A^*, which are moreover classical symbols. By inverse Fourier transform on the fibres, each symbol $a \in S^m(A^*)$ defines a distribution $\mathcal{F}^{-1}_{fib}(a)$ on A that is conormal to M. By restriction, $\mathcal{F}^{-1}_{fib}(a)$ defines a distribution on TM_0 conormal to M_0. We fix a metric on A which then defines a compatible metric. We denote by exp the (geodesic) exponential map associated to this metric (yielding $\exp_x : T_xM_0 \to M_0$ for each $x \in M_0$). Now for some $r > 0$, let

$$\Phi : (TM_0)_r \to V_r \subset M_0 \times M_0, v \in (T_xM_0)_r \mapsto (x, exp_x(-v))$$

be the diffeomorphism given by the Riemann-Weyl fibration, where $(TM_0)_r$ are the vectors with norm less than r, V_r is an open neighborhood of the diagonal $M_0 \cong \Delta_{M_0} \subset M_0^2$, and $r > 0$ is less than the injectivity radius of M_0, which is known to be positive. Fix a smooth function χ, with $\mathrm{supp}\,\chi \in A_r$ and $\chi = 1$ on a neighborhhod of the zero section of A, which is identified with M. For $a \in S^m(A^*)$, define a distribution on M_0^2, conormal to M_0 by

$$q_\chi(a) := \Phi_*(\chi \mathcal{F}^{-1}_{fib}(a)). \tag{18}$$

Let $a_\chi(D)$ denote the operator on M_0 with Schwartz kernel q_χ. Then $a_\chi(D)$ is a properly supported (if $r < \infty$) pseudodifferential operator on M_0.

For each $X \in \mathcal{V} = \Gamma(A)$, let $\psi_X : \mathcal{C}^\infty_c(M) \to \mathcal{C}^\infty_c(M)$ be the operator induced by the global flow $\Psi_X : \mathbb{R} \times M \to M$ by evaluation at 1.

Definition 2.2. The space $\Psi^m_{\mathcal{V}}(M_0)$ of *pseudodifferential operators generated by the Lie structure at infinity* $(M, \mathcal{V})$ is the linear space of operators $\mathcal{C}^\infty_c(M_0) \to \mathcal{C}^\infty_c(M_0)$ generated by $a_\chi(D)$, with $a \in S^m(A^*)$ and $b_\chi(D)\psi_{X_1}\cdots\psi_{X_k}$, with $b \in S^{-\infty}(A^*)$ and $X_1, \cdots, X_k \in \mathcal{V} = \Gamma(A)$.

We define similarly the space $\Psi^m_{\mathcal{V}}(M_0; E, F)$ of pseudodifferential operators acting between sections of vector bundles E, F over M.

As for the usual algebras of pseudodifferential operators, we have the following basic property of the principal symbol (Proposition 2.6 [1]): the principal symbol establishes isomorphisms

$$\sigma_m : \Psi^m_{\mathcal{V}}(M_0)/\Psi^{m-1}_{\mathcal{V}}(M_0) \to S^m(A^*)/S^{m-1}(A^*) \cong \mathcal{C}^\infty(S^*A).$$

We note that the set of all $a_\chi(D)$, with $a \in S^\infty(A^*)$ is *not* closed under composition of operators, that is why we consider extra operators in $\Psi^{-\infty}(M_0)$. To show that $\Psi^m_{\mathcal{V}}(M_0)$ is indeed closed under composition, results from the following section are needed.

2.2. **Operators on groupoids.** In order to obtain algebraic properties and, in particular, Fredholm criteria, an important result is that $\Psi^m_{\mathcal{V}}(M_0)$ can be recovered from an algebra of pseudodifferential operators on a suitable groupoid integrating A. We review the main definitions of the theory of pseudodifferential operators on groupoids, for the benefit of the reader (see [32, 39]).

For a Lie groupoid $\mathcal{G}$ with space of units given by a manifold with corners M, with $d, r : \mathcal{G} \to M$ the domain and range maps, $P = (P_x) \in \Psi^m(\mathcal{G})$ is defined as a smooth family of pseudodifferential operators on the fibres $\mathcal{G}_x := d^{-1}(x)$, $x \in M$, which is right-invariant, that is, $U_g P_{d(g)} = P_{r(g)} U_g$ where $U_g : C^\infty(\mathcal{G}_{d(g)}) \to C^\infty(\mathcal{G}_{r(g)}), U_g(f)g' := f(g'g)$. Recall that the definition of a Lie groupoid requires the sets $\mathcal{G}_x := d^{-1}(x)$ to be smooth manifolds (no corners). We also assume that this family is uniformly supported, in that

$$\mathrm{supp}(P) = \overline{\cup_x \mu(\mathrm{supp}(K_x))} \subset \mathcal{G}$$

is compact, where $\mu(g, h) = gh^{-1}$ and K_x denotes the Schwartz kernel of P_x (a distribution on $\mathcal{G}_x \times \mathcal{G}_x$). In this case, each P_x is properly supported, so that the composition gh^{-1} is well defined. Moreover, P acts on $C^\infty(\mathcal{G})$. Let $T^d\mathcal{G} = \ker d^* = \cup T_x\mathcal{G}_x$ be the d-vertical tangent bundle and denote by $A(\mathcal{G}) := \left(T^d\mathcal{G}\right)_M$ the *Lie algebroid* of $\mathcal{G}$ and $S^*A(\mathcal{G}) := (A(\mathcal{G})^* \backslash 0)/\mathbb{R}^*_+$ its cosphere bundle. Let us fix a metric on A. This choice defines a principal symbol map $\sigma_m : \Psi^m(\mathcal{G}) \to \mathcal{C}^\infty_c(S^*A(\mathcal{G}))$, which is surjective, with kernel $\Psi^{m-1}(\mathcal{G})$. One can define similarly operators acting between sections of vector bundles: if E is a vector bundle over the space of units M, then $\Psi^m(\mathcal{G}, r^*E)$ is well-defined as above.

For each $x \in M$, we consider the *regular representation* π_x of $\Psi^\infty(\mathcal{G})$ on $\mathcal{C}^\infty(\mathcal{G}_x)$ defined as $\pi_x(P) := P_x$. When restricted to order zero operators, this is a bounded $*$-representation, for all x. For $P \in \Psi^0(\mathcal{G})$, let

$$\|P\|_r := \sup_{x \in M} \|\pi_x(P)\| \tag{19}$$

be the *reduced C^*-norm*. We shall also need the *full C^*-norm* defined as

$$\|P\| := \sup_{\rho} \|\rho(P)\|,$$

where ρ ranges through bounded $*$-representation of $\Psi^0(\mathcal{G})$ such that for $T \in \Psi^{-\infty}(\mathcal{G})$, $\rho(T) \leq \|T\|_1$, with $\| \ \|_1$ defined by integrating the Schwartz kernels over the fibres (see [32] for the precise definitions). Endowing $\Psi^0(\mathcal{G})$ with the full norm $\| \ \|$, we have that the principal symbol extends to a bounded $*$-homomorphism

$$\sigma_0 : \overline{\Psi^0(\mathcal{G})} \to \mathcal{C}_0(S^*A(\mathcal{G})), \tag{20}$$

surjective, with kernel $C^*(\mathcal{G}) := \overline{\Psi^{-\infty}(\mathcal{G})})$. (A similar result holds for the reduced norm.)

If $Y \subset M$ is an invariant subset (that is, $d^{-1}(Y) = r^{-1}(Y)$), then $\mathcal{G}_Y := d^{-1}(Y)$ is also a continuous family groupoid, with units Y and there is a well-defined restriction map $\mathcal{R}_Y : \Psi^m(\mathcal{G}; E) \to \Psi^m(\mathcal{G}_Y; E_Y)$. In this case, Lemma 3 in [32] gives that the following sequence is exact:

$$0 \longrightarrow C^*(\mathcal{G}_{M\setminus Y}) \longrightarrow \overline{\Psi^0(\mathcal{G})} \xrightarrow{(\sigma_0, \mathcal{R}_Y)} \mathcal{C}(S^*A(\mathcal{G})) \times_{\mathcal{C}(S^*A(\mathcal{G})_Y)} \overline{\Psi^0(\mathcal{G}_Y)} \longrightarrow 0, \tag{21}$$

where the fibered product $\overline{\Psi^0(\mathcal{G}_Y)} \times_{\mathcal{C}_0(S^*A(\mathcal{G})_Y)} \mathcal{C}_0(S^*A(\mathcal{G}))$ is defined as the algebra of pairs $(Q, f) \in \overline{\Psi^0(\mathcal{G}_Y)} \times \mathcal{C}_0(S^*A(\mathcal{G}))$ such that $\sigma_0(Q) = f_{|S^*A(\mathcal{G})_Y}$.

If $M_0 = int(M)$ is an invariant subset, one can define the so-called *vector representation* π_{M_0}, which associates to $P \in \Psi^0(\mathcal{G})$ a pseudodifferential operator $\pi_{M_0}(P) : \mathcal{C}_c^\infty(M_0) \to \mathcal{C}_c^\infty(M_0)$ by the formula $\pi_{M_0}(P)u = u_0$, with $P(u \circ r) = u_0 \circ r$ [34]. Recall that a Lie groupoid is called *d-connected* if all the sets $\mathcal{G}_x := d^{-1}(x)$ are connected. If $A \to M$ is a Lie algebroid on M, we say that $\mathcal{G}$ integrates A if $A(\mathcal{G}) = A$. We shall need Theorem 3.3 from [1], which gives that

Theorem 2.3. *Let $(M, \mathcal{V})$ be a Lie manifold with Lie algebroid A and $\mathcal{G}$ be a d-connected groupoid over M integrating A. Then $\Psi^m_{\mathcal{V}}(M_0) \cong \pi_{M_0}(\Psi^m(\mathcal{G}))$.*

The right-hand-side is well defined since, as we shall see next, one can always assume that M_0 is an invariant subset of such $\mathcal{G}$. In particular, it follows that the classes $\Psi^m_{\mathcal{V}}(M_0)$ define a filtered algebra on $\Psi^\infty_{\mathcal{V}}(M_0)$.

The problem of integrating Lie algebroids was solved in [20], though for our purposes, the results in [38] suffice. Namely, M_0 and ∂M form an A-invariant stratification of M, so it follows from the glueing theorem in [38] that it suffices to integrate along these strata. Since the anchor map ρ is a diffeomorphism over the interior, we can take the d-connected groupoid $\mathcal{G}$ integrating A to coincide with the pair groupoid over the interior, meaning that $\mathcal{G}_{M_0} \cong M_0 \times M_0$, in case M_0 is connected. (The general case of non-connected M_0 can be reduced to the connected case by taking the compactification of each connected component.) It then follows from [38] that, if $\mathcal{G}_{\partial M}$ is a groupoid integrating $A_{\partial M}$, then

$$\mathcal{G} = \mathcal{G}_{M_0} \sqcup \mathcal{G}_{\partial M} \cong (M_0 \times M_0) \cup \mathcal{G}_{\partial M}, \tag{22}$$

has the structure of a differentiable groupoid with Lie algebroid A. We see that M_0 is indeed an invariant subset, and moreover, since $\mathcal{G}_{M_0}$ is the pair groupoid, one has that $C^*(\mathcal{G}_{M_0}) \cong \mathcal{K}(L^2(M_0))$, the isomorphism being induced either by the vector representation π_{M_0} or by π_x, $x \in M_0$, noting that these representations are equivalent through the isometry $r : \mathcal{G}_x \to M_0$. In particular, the vector representation π_{M_0} is bounded. Fredholm criteria now follow from the exact sequence (21) as in [32] (Theorem 4).

From now on we shall assume that $\mathcal{G}$ is a d-connected Lie groupoid integrating the Lie algebroid $A \to M$ defined by a Lie manifold $(M, \mathcal{V})$. We shall also assume that the vector representation π_{M_0} is injective on $\overline{\Psi^0(\mathcal{G})}$. In particular, $\Psi^0(\mathcal{G}) \cong \Psi^0_{\mathcal{V}}(M_0)$. Moreover, since $C^*(\mathcal{G}) \subset \overline{\Psi^0(\mathcal{G})}$ and π_{M_0} factors through the reduced C^*-algebra of $\mathcal{G}$, so that $\pi_{M_0}(C^*_r(\mathcal{G})) = \pi_{M_0}(C^*(\mathcal{G}))$, we hence obtain that $\mathcal{G}$ is amenable, in that the reduced and full norms coincide.

We shall use the isomorphism above to carry to $\Psi^m_{\mathcal{V}}(M_0)$ all concepts defined for $\Psi^m(\mathcal{G})$. At the level of symbols, we have $\sigma_m(P) = \sigma_m(\pi_{M_0}(Q)) = \sigma_m(Q)$ on M_0, for any $P \in \Psi^m_{\mathcal{V}}(M_0)$. We shall also need the map of restriction to the boundary for operators on $(M, \mathcal{V})$

$$\sigma_\partial : \overline{\Psi^0_{\mathcal{V}}(M_0)} \to \overline{\Psi^0(\mathcal{G}_{\partial M})}, \quad P \mapsto \mathcal{R}_\partial(Q) = Q|_{\partial M},$$

where $\pi_{M_0}(Q) = P$ and $\mathcal{R}_\partial : \Psi^0(\mathcal{G}) \to \Psi^0(\mathcal{G}_{\partial M})$ is restriction to the boundary.

Proposition 2.4. *Let $(M, \mathcal{V})$ be a Lie manifold with Lie algebroid A and $\mathcal{G}$ be a d-connected groupoid as in* (22) *satisfying $A(\mathcal{G}) \simeq A$. Assume that the representation π_{M_0} is injective on $\overline{\Psi^0(\mathcal{G})}$, as above. Then*

$$\overline{\Psi^0(\mathcal{G})}/\mathcal{K} \cong \mathcal{C}(S^*A) \times_{\mathcal{C}(S^*A_{\partial M})} \overline{\Psi^0(\mathcal{G}_{\partial M})}$$
$$:= \{(a, Q) \in \mathcal{C}_0(S^*A) \times \overline{\Psi^0(\mathcal{G}_{\partial M})}, a|_{\partial M} = \sigma_0(Q) \in \mathcal{C}(S^*A_{\partial M})\}$$

and $P \in \overline{\Psi^0_{\mathcal{V}}(M_0)}$ is Fredholm if, and only if, it is elliptic and $\sigma_\partial(P)$ is invertible in $\overline{\Psi^0(\mathcal{G}_{\partial M})}$.

Proof. Since π_{M_0} is injective and $\pi_{M_0} C^*(\mathcal{G}_{M_0}) \cong \mathcal{K}(L^2(M_0))$, we have the induced representation $\pi' : \overline{\Psi^0(\mathcal{G})}/C^*(\mathcal{G}_{M_0}) \to \mathcal{B}(L^2(M_0))/\mathcal{K}$, which is also injective. Hence, $P = \pi_{M_0}(Q)$ is Fredholm if, and only if the class of Q is invertible in $\overline{\Psi^0(\mathcal{G})}/C^*(\mathcal{G}_{M_0})$. The result follows from (21). □

Moreover, the amenability of $\mathcal{G}$ yields that the restriction $\mathcal{G}_{\partial M}$ is also amenable [43] Prop. 3.7). In this case, $\rho := \Pi_{x \in \partial M} \pi_x$ is an injective representation of $\Psi^0(\mathcal{G}_{\partial M})$ and $\sigma_\partial(P)$, as above, is invertible if, and only if, $\sigma_\partial(P)_x = Q_x$ is invertible for all $x \in \partial M$, with $\pi_{M_0}(Q) = P$. (The same is true also for $\Psi^\infty(\mathcal{G}_{\partial M})$, since if $\rho(P) = 0$, then $\rho(P(1 + P^*P)^{-1/2}) = 0$.)

Elliptic operators P with invertible $\sigma_\partial(P)$ are sometimes called *fully elliptic* and the algebra $\Psi^0(\mathcal{G}_{\partial M})$ is the so-called *indicial algebra*. If π_{M_0} is not injective for some $x \in M_0$, then we only have a sufficient condition for Fredholmness.

To finish this section, we prove a result that will later enable us to compute the index of operators with order $m > 0$ from the index of order 0 operators.

Lemma 2.5. *Let $Q \in \Psi^m_{\mathcal{V}}(M_0; E)$ and $P := Q(1 + Q^*Q)^{-1/2}$. Then $P \in \overline{\Psi^0_{\mathcal{V}}(M_0; E)}$*

Proof. Let $\mathcal{G}$ be the canonical groupoid integrating $(M, \mathcal{V})$. It follows from groupoid calculus applied to $\Psi^0(\mathcal{G})$, more precisely from Theorem 7.2 in [34], that if $L \in \Psi^{2m}(\mathcal{G})$ is such that $L \geq 1$ and $\sigma_{2m}(L) > 0$ then $SL^{-1/2} \in \overline{\Psi^0(\mathcal{G})}$, for any $S \in \Psi^m(\mathcal{G})$. From Theorem 2.3, let $R \in \Psi^m(\mathcal{G})$ be such that $\pi_{M_0}(R) = Q$, with π_{M_0} the vector representation. Then

$$1 + Q^*Q = 1 + \pi_{M_0}(R)^*\pi_{M_0}(R) = \pi_{M_0}(1 + R^*R)$$

and we can apply the result above to $1 + R^*R$ to obtain $R(1 + R^*R)^{-1/2} \in \overline{\Psi^0(\mathcal{G})}$. Hence $\pi_{M_0}(R(1 + R^*R)^{-1/2}) \in \overline{\Psi_{\mathcal{V}}{}^0(M_0)}$. It follows from the definitions that

$$\pi_{M_0}((1 + R^*R)^{-1/2}) = \pi_{M_0}(1 + R^*R)^{-1/2},$$

so that $P = Q(1 + Q^*Q)^{-1/2} \in \overline{\Psi^0_{\mathcal{V}}(M_0)}$. □

Note that if P and Q are as in the lemma above (Lemma 2.5), $\sigma_m(Q)$ and $\sigma_0(P)$ are homotopic as sections of S^*A. Moreover, if we define the map of restriction to the boundary $\sigma_\partial : \Psi^m_{\mathcal{V}}(M_0; E) \to \Psi^m(\mathcal{G}_{\partial M}; r^*E)$ given, as before, by $\sigma_\partial(Q) := \mathcal{R}_\partial(R)$, with $\pi_{M_0}(R) = Q$, then it follows from the proof that

$$\sigma_\partial(P) = \sigma_\partial(Q(1 + Q^*Q)^{-1/2}) = \sigma_\partial(Q)(1 + \sigma_\partial(Q)^*\sigma_\partial(Q))^{-1/2},$$

hence $\sigma_\partial(P)$ is invertible if, and only if, $\sigma_\partial(Q)$ is. We say that $Q \in \Psi^m_{\mathcal{V}}(M_0; E)$ is *fully elliptic* if, and only if, P is. In that case, P is Fredholm and Q will also be Fredholm, in the setting of unbounded operators, with $\mathrm{ind}(Q) = \mathrm{ind}(P)$ (see Section 3.1).

2.3. **The asymptotically commutative case.** In this subsection, we prove an index formula for certain classes of pseudodifferential operators on Lie manifolds whose associated groupoids are such that the restrictions at the boundary yield bundles of commutative Lie groups. The main point is giving conditions that yield commutativity of the algebra $\Psi^0(\mathcal{G}_{\partial M})$, using the notation of the previous section, so that Fredholmness depends on invertibility in an algebra of functions, thus reducing to the setting considered in Section 1.3. This is known to hold for the scattering and double-edge calculus [32, 33, 34, 36, 41]. The dimension of M is denoted by n, as before. Recall that in this section, we do not assume n to be even.

Definition 2.6. Let $(M, \mathcal{W})$ be a connected Lie manifold with Lie algebroid $\pi : A_{\mathcal{W}} \to M$ with the property that any $X \in \mathcal{W}$ vanishes at the boundary ∂M (that is, on any face of the boundary) and the resulting Lie algebras $A_{\mathcal{W},x} := \pi^{-1}(x)$ are commutative. A Lie manifold $(M, \mathcal{W})$ with this property will be called an *asymptotically commutative* Lie manifold, and $\mathcal{W}$ will be called *commutative at infinity*.

(We reserve the notation $\mathcal{W}$ for asymptotically commutative structural Lie algebras of vector fields, whereas $\mathcal{V}$ will denote a general such structural Lie algebra of vector fields.)

Let $(M, \mathcal{W})$ be an asymptotically commutative Lie manifold. Then a groupoid integrating $A_{\mathcal{W}}|_{\partial M}$ is $A_{\mathcal{W}}|_{\partial M}$ itself (since the commutative Lie algebra $\mathbb{R}^n$ identifies with itself via the exponential map). According to [38], there will be a unique Lie manifold structure on the disjoint union

$$\mathcal{G} := (M_0 \times M_0) \cup A_{\mathcal{W}}|_{\partial M} \tag{23}$$

such that $\mathcal{G}$ is a Lie groupoid integrating $A = A_{\mathcal{W}}$. Thus any Lie algebroid associated to an asymptotically commutative Lie manifold has a canonical Lie groupoid integrating it. Let $\overline{A}$ be the sphere bundle obtained by radial compactification of the fibres of A.

Proposition 2.7. *Assume* $(M, \mathcal{W})$ *is an asymptotically commutative Lie manifold and let* $\mathcal{G}$ *be the canonical Lie algebroid integrating it, as in Equation* (23)*. Then* $\Psi^0(\mathcal{G}_{\partial M})$ *is commutative and*

$$\overline{\Psi^0(\mathcal{G}_{\partial M})} \cong \mathcal{C}(\overline{A}|_{\partial M}).$$

Proof. It follows from (23) that the algebra of pseudodifferential operators on $\mathcal{G}_{\partial M}$ coincides with $\Psi^0(A_{\partial M})$, that is, with the algebra of continuous families of (P_x) of translation invariant pseudodifferential operators P_x acting on the fibers $(A_{\partial M})_x$ of $A_{\partial M}$, $x \in \partial M$.

Now, the pseudodifferential operators of order zero on a vector space V that are translation invariant coincide with convolution operators with functions whose Fourier transform is in

$$\widetilde{S^0}(V) = \{p \in \mathcal{C}^\infty(V) \colon p(y,\xi) := p(\xi) \in S^0(T^*V)\}$$

(symbols of order zero that are independent of y). The algebra of convolution operators is commutative, so it follows straight away that $\Psi^0(A_{\partial M})$ is commutative. (In particular, the reduced and full C^*-norms coincide.) Moreover, one can check that $\overline{\widetilde{S^0}(V)} \cong \mathcal{C}(\overline{V})$, with $\overline{V}$ the radial compactification of V. Hence, $\overline{\Psi^0(A_{\partial M})} \cong \mathcal{C}(\overline{A}|_{\partial M})$, since there is a isomorphism between elements of $\Psi^0(A_{\partial M})$ and continuous families in $\widetilde{S^0}(A_x)$, which is bounded with respect to the reduced, hence the full, norm. This proves $\overline{\Psi^0(\mathcal{G}_\partial))} \cong \mathcal{C}(\overline{A}|_{\partial M})$, as claimed. □

Note that it follows from the proof that the isomorphism above is really given by the total symbol, as in (4), of the indicial boundary operator. For order $m > 0$ operators, we have $\Psi^m(\mathcal{G}_{\partial M}) = \Psi^m(A|_{\partial M}) \cong \widetilde{S^m}(A|_{\partial M}) \subset \mathcal{C}^\infty(A|_{\partial M})$, the isomorphism being again given by the total symbol, that is, including the lower order terms of the symbol. (This total symbol is defined since the resulting operators on the fibers of $A \to \partial M$ are translation invariant, and hence they are convolution operators. The total symbol is simply the Fourier transform of the resulting convolution distributions.)

As in Section 1.3, Equation (13) consider $\Omega := \partial(\overline{A}) = (S^*A) \cup \overline{A}|_{\partial M}$ such that $\mathcal{C}(\Omega) = \{(f,g) \in \mathcal{C}(S^*A) \oplus \mathcal{C}(\overline{A}|_{\partial M}), f = g \text{ on } S^*A_{\partial M}\}$.

Define the *boundary symbol* for operators on $(M, \mathcal{W})$ by

$$\sigma_\partial : \overline{\Psi^0_{\mathcal{W}}(M_0; E)} \to \mathcal{C}(\overline{A}|_{\partial M}) \tag{24}$$

as the map of restriction to the boundary composed with the isomorphism given by the previous proposition. For $P \in \Psi^m_{\mathcal{W}}(M_0; E)$, the boundary symbol is just given by the total symbol of $\mathcal{R}_\partial(Q) = Q|_{\partial M} \in \Psi^0(A_{\partial M})$, with $\mathcal{R}_{\partial M} : \Psi^m(\mathcal{G}, r^*E) \to \Psi^m(A_{\partial M}, r^*E_{\partial M})$ the restriction map and $\pi_{M_0}(Q) = P$.

Moreover, it follows from (21) that $\mathcal{C}(\Omega)$ is the recipient of full symbols of pseudodifferential operators on M, since $\mathcal{C}(\overline{A}|_{\partial M}) \times_{\mathcal{C}_0(S^*A_{\partial M})} \mathcal{C}_0(S^*A) = \mathcal{C}(\partial\overline{A}) = \mathcal{C}(\Omega)$. We have then a map

$$\sigma_{full} := (\sigma_0, \sigma_\partial) : \overline{\Psi^0_{\mathcal{W}}(M; E)} \to \mathcal{C}(\Omega), \tag{25}$$

which is surjective, continuous and a $*$-algebra morphism. We will see in the next proposition that $\mathcal{K} \subset \ker \sigma_{full}$, so it follows that $\Omega = \partial(\overline{A})$ is a comparison space for $\overline{\Psi^0_{\mathcal{W}}(M; E)}$ (see Equation (14)), and hence the results from Section 1.3 apply.

Proposition 2.8. *Assume $(M, \mathcal{W})$ is an asymptotically commutative Lie manifold and let $\mathcal{G}$ be the canonical Lie groupoid integrating it, as in Equation (23). Then π_{M_0} is injective on $\overline{\Psi^0(\mathcal{G})}$, and hence the following sequence is exact.*

$$0 \longrightarrow \mathcal{K}(M; E) \longrightarrow \overline{\Psi^0_{\mathcal{W}}(M; E)} \xrightarrow{(\sigma_\partial, \sigma_0)} \mathcal{C}(\Omega) \longrightarrow 0. \tag{26}$$

In particular, an operator $P \in \overline{\Psi^0_{\mathcal{W}}(M; E)}$ is Fredholm if, and only if, it is fully elliptic, meaning that $\sigma_{full}(P) = (\sigma_0(P), \sigma_\partial(P)) \in \mathcal{C}(\Omega)$ is invertible.

Proof. The second part will follow from the first part using Proposition 2.4, so we concentrate on proving the injectivity of π_{M_0}. Let I be the kernel of π_{M_0}. We want to show that $I = \{0\}$. We have that π_{M_0} is injective on the subalgebra of compact operators of $\Psi^0_{\mathcal{W}}(M, E)$, so $I \cap \mathcal{K} = 0$. It follows that $(\sigma_0, \sigma_{\partial M})$ is injective on I, since it has kernel $\mathcal{K}$.

Let $P \in I \subset \Psi^0_{\mathcal{W}}(M, E)$. We can recover the principal symbol of P from its action on M_0 [2, 28, 34] so we can assume $m < 0$. By replacing P with a power of P^*P, we can assume that $m < -n$. The Fourier transform (as in Equation (18)) then allows us to recover the boundary symbol of P since for x approaching the boundary, the exponential map increases its radius of injectivity (so the cutoff χ will affect less and less the kernel of the resulting operator). This shows that π_{M_0} is injective on $\overline{\Psi^0(\mathcal{G})}$. $\square$

In this case, let $[\sigma_0(P)] \in K^0(TM_0)$ denote the K^0-theory class associated to P, as in (16), and $[\sigma_{full}(P)] := [(\sigma_\partial(P), \sigma_0(P))] \in K_1(\mathcal{C}(\Omega)) \cong K^1(\Omega)$ denote the class in K^1. As before, let $Td(T_\mathbb{C}M)$ denote the Todd class of the complexified tangent bundle of M, and $\pi : \overline{T^*M} \to M$. Also, we denote by $\pi_\Omega : \Omega = \partial(\overline{A}) \to M$ the natural projection. From Theorem 1.8 it finally follows:

Theorem 2.9. *Let $(M, \mathcal{W})$ be an asymptotically commutative Lie manifold manifold with Lie algebroid A, $\Omega := \partial(\overline{A})$, and let $P \in \overline{\Psi^0_{\mathcal{W}}(M; E)}$ be an elliptic operator with $\sigma_\partial(P)$ invertible in $\mathcal{C}(\overline{A}_{\partial M})$. Then,*

$$\operatorname{ind}(P) = (-1)^n \operatorname{ch}_0[\widetilde{\sigma}_{full}(P)]\pi^* Td(T_\mathbb{C}M)[TM_0] = (-1)^n \operatorname{ch}_1[\sigma_{full}(P)]\pi^*_\Omega Td(T_\mathbb{C}M)[\Omega],$$

where $[\widetilde{\sigma}_{full}(P)] \in K^0(TM_0)$ is defined using Lemma 1.7.

Our main example of an asymptotically commutative Lie manifold $(M, \mathcal{W})$ is obtained as follows. Let $(M, \mathcal{V})$ be a Lie manifold and let x_k be boundary defining functions of the hyperfaces of M. Choose $a_k \in \mathbb{N} = \{1, 2, \ldots\}$. Then, as in the Equation (3), we introduce

$$\mathcal{W} := f\mathcal{V}, \text{ with } f := \Pi x_k^{a_k},$$

is also a structural Lie algebra of vector fields, since it is closed for Lie brackets, and a finitely generated, projective $\mathcal{C}^\infty(M)$–module. Hence $(M, \mathcal{W})$ is a Lie manifold that is easily seen to be asymptotically commutative.

The previous result extends the known index formulas for the scattering calculus on manifolds with boundary, where $\mathcal{V}_{sc} := x\mathcal{V}_b$, with x is a boundary defining function and $\mathcal{V}_b$ is the Lie algebra of vector fields tangent to the boundary, and for the double-edge calculus, where $\mathcal{V}_{de} = x\mathcal{V}_e$, with $\mathcal{V}_e$ the edge vector fields induced by a fibration of the boundary [26, 30, 33, 36]. Moreover, the index formula above can be proved in the same way considering families of pseudodifferential operators over a compact base space B (the index now takes values in $K^0(B)$) using a generalization of the Atiyah-Singer index theorem for families of asymptotically multiplication operators. In this sense, Theorem 2.9 yields the result in [30] for families of scattering pseudodifferential operators.

In the next section, we will apply the index formula above to compute the index of perturbed Dirac operators on *general* Lie manifolds.

3. Perturbed Dirac operators

Throughout this section, we let M_0 be a non-compact, *even dimensional* manifold M_0, which, as before, is assumed to be the interior of a Lie manfold $(M, \mathcal{V})$. We fix a set $\{x_k\}$ of defining functions of M and let

$$f := \Pi x_k^{a_k}, \ a_k \in \mathbb{N}, \tag{27}$$

(so $a_k > 0$). We consider in this section a Dirac operator $\not{D}$ coupled with a potential V, that is, an operator of the form

$$T = \not{D} + V := \not{D}\widehat{\otimes}1 + 1\widehat{\otimes}V \tag{28}$$

on compactly supported sections of some vector bundles defined on M_0. By a *potential* we shall always mean an odd, self-adjoint endomorphism of a $\mathbb{Z}_2$-graded vector bundle over M_0. An operator T of this type with will be called a *Callias-type operator*. (More precisely, T is the closure of $T\widehat{\otimes}1 + 1\widehat{\otimes}V$.) We assume the potential V to be of the form

$$V := f^{-1}V_0 = \Pi x_k^{-a_k} V_0,$$

where V_0 extends to a smooth function on M, *invertible* at the boundary. In particular, the potential V is *unbounded*.

We apply the results of the previous section to give a cohomological formula for the index of $T^+ := (\not{D} + V)^+$. The main point is to reduce the calculation of the index of T^+ to the case of a Dirac operator coupled with a *bounded potential* on the asymptotically commutative Lie manifold $(M, \mathcal{W})$ defined by $\mathcal{W} := f\mathcal{V}$, and show that the index can be obtained from Theorem 2.9. More precisely, we shall show that

$$\operatorname{ind}(T^+) = \operatorname{ind}(Q) \ \text{ for } \ Q := f^{1/2}T^+f^{1/2} \in \Psi^1_{\mathcal{W}}(M; F_0, F_1), \ \mathcal{W} := f\mathcal{V}, \tag{29}$$

for suitable vector bundles F_0 and F_1. We then use that

$$P := Q(1 + Q^*Q)^{-1/2} \in \overline{\Psi^0_{\mathcal{W}}(M; F_0, F_1)}$$

also satifies $\operatorname{ind}(P^+) = \operatorname{ind}(Q^+)$. Finally, we show that $\operatorname{ind}(P^+)$, and hence also $\operatorname{ind}(T^+) = \operatorname{ind}(P^+)$, can be computed using Theorem 2.9.

3.1. **Dirac and Callias operators.** Let W and E be $\mathbb{Z}_2$-graded vector bundles over M. We endow $W \otimes E$ with the usual grading and denote by $W\widehat{\otimes}E$ the resulting $\mathbb{Z}_2$-graded vector bundle, namely,

$$(W\widehat{\otimes}E)^+ = (W^+ \otimes E^+) \oplus (W^- \otimes E^-) \ \text{ and } \ (W\widehat{\otimes}E)^- = (W^- \otimes E^+) \oplus (W^+ \otimes E^-).$$

If $V \in \operatorname{End}(E)$ is an endomorphism, then it acts on $C^\infty(E)$ as a (pseudo)differential operator of order 0.

Definition 3.1. An operator $T : \mathcal{C}^\infty_c(M_0; W\widehat{\otimes}E) \to \mathcal{C}^\infty_c(M_0; W\widehat{\otimes}E)$ is said to be a *Callias-type pseudodifferential operator* on the Lie manifold $(M, \mathcal{V})$ if

$$T := D + V := D\widehat{\otimes}1 + 1\widehat{\otimes}V.$$

where $D \in \Psi^m_{\mathcal{V}}(M, W)$, $m > 0$, is an odd, symmetric, elliptic operator and $V \in \operatorname{End}(E|_{M_0})$ is odd and self-adjoint and invertible outside a compact set. We refer to V as a *potential*. We shall also assume our potential V to be invertible outside a compact subset of M_0. The closure of an operator of the form $T = D + V$ will also be called a *Callias-type operator*.

When D is the (generalized) Dirac operator, these operators are also called Dirac-Schrödinger operators and were first considered by Callias (in the odd dimensional Euclidean space [16]). See also [14, 15, 23, 22] and references therein for more results on index theory of Dirac-Schrödinger and Callias type operators on even-dimensional manifolds.

Remark 3.2. On odd dimensional manifolds, the Callias-type operators are of the form $\not{D} + iV$, where V is self-adjoint and invertible at infinity. See [5, 6, 12, 13, 16, 21, 30, 42] for more on the index of Calias type operators in the odd-case.

Recall that a symmetric (hence closable) operator T is *essentially self-adjoint* if its closure is self-adjoint, that is, if $< Tx, y >=< x, Ty >$, for all $x, y \in \mathcal{D}(T) = \mathcal{D}(T^*)$. (We shall always denote the minimal closure of an operator by the same letter.)

In the following lemma, we assume that the potential V_0 extends to M, in particular, it is bounded. (We will prove such a result for an unbounded potential in Section 3.4.)

Lemma 3.3. *Let $D \in \Psi^m_{\mathcal{V}}(M, W)$, $m > 0$, be an odd, symmetric, elliptic operator. Assume that V_0 extends to a smooth function on M, as before, then the Callias-type operator $T = D + V_0 \in \Psi^m_{\mathcal{V}}(M_0; W \otimes E)$ is elliptic and essentially self-adjoint on $\mathcal{C}^\infty_c(M_0; W \otimes E)$.*

Proof. Ellipticity follows from $\sigma_m(T) = \sigma_m(D)$. The fact that T is essentially self-adjoint follows, for instance, from [34] (Theorem 7.1) which yields that, with $m > 0$, a (possibly unbounded) symmetric, elliptic operator in $\Psi^m_{\mathcal{V}}(M; W \otimes E)$ is essentially self-adjoint, identifying $\Psi^m_{\mathcal{V}}(M; W \otimes E) = \pi_{M_0}(\Psi^m(\mathcal{G}; r^*(W \otimes E)))$, as in Theorem 2.3. □

We shall work with unbounded Fredholm operators. It will then be useful to recall the way they are introduced. Let T be a possibly unbounded operator with domain $\mathcal{D}(T)$ and codomain H. We shall always replace T by its closure, so assume T is closed and endow $\mathcal{D}(T)$ with the graph norm. Then T is Fredholm if, by definition, the induced bounded operator $T : \mathcal{D}(T) \to H$ is Fredholm (in the usual sense of having finite dimensional kernel and cokernel). In particular, a pseudodifferential operator T_1 acting between sections of E_0 with range sections of E_1 is Fredholm if, and only if, $T_2 := T_1(1 + T_1^* T_1)^{-1/2}$ is a Fredholm operator and, in this case, $\operatorname{ind}(T_1) = \operatorname{ind}(T_2)$.

We are interested in computing the index of

$$T^+ = (D + V)^+ : \mathcal{C}^\infty_c(M_0; (W \widehat{\otimes} E)^+) \to \mathcal{C}^\infty_c(M_0; (W \widehat{\otimes} E)^-), \tag{30}$$

which we shall prove to be Fredholmness between suitable Sobolev spaces.

Note that, with respect to the grading, we can write

$$T^+ = \begin{pmatrix} D^+ \otimes 1 & -1 \otimes V^- \\ 1 \otimes V^+ & D^- \otimes 1 \end{pmatrix}. \tag{31}$$

Most of our results work for general odd, elliptic, positive pseudodifferential operators $D \in \Psi^m_{\mathcal{V}}(M; E)$. However, for simplicity and because this is the most useful case in applications, we shall mainly be interested in the case when D is a generalized Dirac operator. Recall that, in any case, the Dirac operators generate all classes in K-homology, so we can always assume D to be a Dirac operator.

3.2. **Dirac operators on Lie manifolds.** We introduce here generalized Dirac operators on Lie manifolds following [3]. Let $(M, \mathcal{V})$ be a *even* dimensional Lie manifold endowed with a *compatible* metric g on M_0 and let W be a Clifford module over M endowed with an A^*-valued connection ∇^W and a Clifford multiplication bundle map $c : A \otimes W \to W$. Recall that a *compatible* metric on M_0 is a metric on TM_0 that extends to $A \to M$. The restrictions of W, c, and ∇^W to M_0 reduce to the classical notions of a Clifford bundle together with an admissible connection [25, 35, 36].

Definition 3.4. The *generalized Dirac operator* $\not{D} : \mathcal{C}^\infty(M; W) \to \mathcal{C}^\infty(M; W)$ associated to W is then defined as the composition

$$\mathcal{C}^\infty(M; W) \xrightarrow{\nabla^W} \mathcal{C}^\infty(M; W \otimes A^*) \xrightarrow{id\otimes\phi} \mathcal{C}^\infty(M; W \otimes A) \xrightarrow{c} \mathcal{C}^\infty(M; W), \tag{32}$$

where $\phi : A^* \to A$ is the isomorphism given by the metric.

Since both the Clifford multiplication c and the A^*-valued connection are $\mathcal{V}$-differential operators, of order 0 and 1, respectively, we have that $\not{D} \in \mathrm{Diff}^1_{\mathcal{V}}(M; W)$. The principal symbol $\sigma_1(\not{D})\xi = ic(\xi) \in \mathrm{End}(W)$ is invertible for any $\xi \neq 0$, and hence $\not{D}$ is elliptic. It follows from classical results that $\not{D}$ with domain $\mathcal{C}^\infty_c(M_0; W) \subset L^2(M_0; W)$ is essentially self-adjoint (*i.e.,* its closure is self-adjoint), since M_0 is complete.

We can also define Dirac operators on groupoids: if $\mathcal{G}$ is a d-connected groupoid integrating $A = A(\mathcal{V})$ then we can consider the Clifford module r^*W and endow $\mathcal{G}$ with an admissible connection $\nabla^{\mathcal{G}} \in \mathrm{Diff}(\mathcal{G}; W, W \otimes A^*)$ such that $\pi_{M_0}(\not{D}^{\mathcal{G}}) = \not{D}$, where $\not{D}^{\mathcal{G}}$ is the associated Dirac operator on $\mathcal{G}$ (see [34] for details).

Assume now that M is even-dimensional and W is $\mathbb{Z}_2$-graded, with the grading given by the chirality operator. Let also E be an Hermitian $\mathbb{Z}_2$-graded vector bundle over M and $V \in \mathrm{End}(E)$ a potential (so odd, self-adjoint). We are interested in computing the index of

$$T^+ = (\not{D} + V)^+ : \mathcal{C}^\infty_c(M_0; (W \otimes E)^+) \to \mathcal{C}^\infty_c(M_0; (W \widehat{\otimes} E)^-). \tag{33}$$

3.3. **The case of bounded potentials.** Let $(M, \mathcal{W})$ be an asymptotically commutative Lie manifold. Recall that in this section we assume that n, the dimension of M, is even.

Let

$$Q := D + V_0 \text{ with } D \in \Psi^m_{\mathcal{W}}(M_0; E \otimes W), \tag{34}$$

where D is an elliptic, symmetric, odd pseudodifferential operator, as in Definition 3.1. Let V_0 be a bounded potential on M_0 that extends to a smooth function on M that is invertible on ∂M (so, in particular, V_0 is odd and symmetric). It follows from Lemma 3.3 that Q is elliptic and essentially self-adjoint on $\mathcal{C}^\infty_c(M_0)$.

We define the total symbol K-theory classes $\sigma_{full}(Q) \in K^1(\Omega)$ and $\widetilde{\sigma}_{full}(Q) \in K^0(TM_0)$ of Q in a similar way to the case of order zero symbols. First, recall that the boundary symbol $\sigma_\partial : \Psi^m_{\mathcal{W}}(M_0; E) \to \mathcal{C}(A|_{\partial M})$ is given by

$$\sigma_\partial(P) = \sigma_m^{tot}(\mathcal{R}_\partial(S)) = \sigma_m^{tot}(S|_{\partial M}),$$

with $\pi_{M_0}(S) = P$, $\mathcal{R}_\partial : \Psi^m(\mathcal{G}) \to \Psi^m(\mathcal{G}_{\partial M})$ is restriction to the boundary and $\sigma_m^{tot}(S_x)$, $x \in \partial M$, is the total symbol of the operator S_x on A_x (including lower order terms).

Lemma 3.5. *Let Q be as in Equation* (34) *and $P := Q(1 + Q^2)^{-1/2}$. Then $P \in \overline{\Psi^0_{\mathcal{W}}(M_0; W \otimes E)}$ is fully elliptic, in the sense that its principal symbol $\sigma_0(P)$ and the boundary symbol $\sigma_\partial(P)$, defined by continuity, are invertible.*

Proof. It follows from Lemma 2.5 that $P \in \overline{\Psi^0_{\mathcal{W}}(M_0; W \otimes E)}$. We have that P is elliptic, since Q is. To understand the boundary operators, since $\mathcal{W}$ is commutative at the boundary, hence $\mathcal{G}_{\partial M} = A_{\partial M}$ is amenable, we only need invertibility on fibres $\mathcal{G}_x = A_x$, $x \in \partial M$ (see the remark after Theorem 2.4). Let $S \in \Psi^m(\mathcal{G}; r^*W)$ be such that $\pi_{M_0}(S) = D$. Therefore, we need to look at the symbol of the operators S_x coupled with the constant potential $V_0(x)$ acting on the fiber A_x, for each $x \in \partial M$. The invertibility of the boundary indicial operator $\sigma_\partial(D + V_0)_x(\xi) = \sigma^{tot}(S_x)\widehat{\otimes}1 + 1\widehat{\otimes}V_0(x)$, $\xi \in A_x$ then follows from the fact that $V_0(x)$ is invertible for each $x \in \partial M$ (noting that $\alpha\widehat{\otimes}1 + 1\widehat{\otimes}\beta \in \mathrm{End}(W_x\widehat{\otimes}E_x)$ is invertible if α or β are invertible.) □

Note that when $D = \not{D}$ is a Dirac operator on $(M, \mathcal{W})$ then, using the notation as above, $S_x = \not{D}_x$ is a Dirac operator on A_x and $\sigma_\partial(\not{D} + V)_x(\xi) = ic(\xi)\widehat{\otimes}1 + 1\widehat{\otimes}V_0(x)$. (This is due to the fact that the restriction of a Dirac operator to the boundary is again a Dirac operator [34].)

The following lemma provides the definitions of the total symbol K-theory classes $\sigma_{full}(Q) \in K^1(\Omega)$ and $\widetilde{\sigma}_{full}(Q) \in K^0(TM_0)$. Let us introduce the K-theory class $[V_0]$ defined by the endomorphism V_0 as usual [7, 29]

$$[V_0] := [E^+, E^-, V_0] \in K^0(M_0) = K^0(M; \partial M) \subset K^0(M).$$

Recall that $\Omega := \partial(\overline{A}) = (S^*A) \cup \overline{A}|_{\partial M}$ (as in Subsection 2.3).

Lemma 3.6. *Let Q and P be as in Lemma 3.5 and define $[\sigma_{full}(Q)] := [\sigma_{full}(P)] \in K^1(\Omega)$ and $[\widetilde{\sigma}_{full}(Q)] := [\widetilde{\sigma}_{full}(P)] \in K^0(TM_0)$. Then*

$$\partial[\sigma_{full}(Q)] = [\widetilde{\sigma}_{full}(Q)]$$

and $[\widetilde{\sigma}_{full}(Q)]$ can be represented by the endomorphism $\sigma_m(D)\widehat{\otimes}1 + 1\widehat{\otimes}V_0$. In particular,

$$[\widetilde{\sigma}_{full}(Q)] = [\sigma_m(D)] \otimes \pi^*[V_0],$$

where $[\sigma_m(D)] \in K^0(TM)$ and $[V_0] \in K^0(M, \partial M)$ are the classes defined by the corresponding morphisms and $\pi : TM \to M$ is the natural projection.

Proof. The relation $\partial[\sigma_{full}(Q)] = [\widetilde{\sigma}_{full}(Q)]$ follows from definitions and from Lemma 1.7. Let us choose a smooth function $\sigma_m \in S^m(A^*)$ such that σ_m represents $\sigma_m(D)$ and on $A^*|_{\partial M}$ it is equal to the total symbol of D. Let then $p = \sigma_m \widehat{\otimes} 1 + 1 \widehat{\otimes} V_0 \in \mathcal{C}^\infty(A^*) = \mathcal{C}^\infty(A)$, where we have used a fixed metric on A to identify A with A^*, as before. From Equation (31) we have that

$$\sigma_0(P) = \sigma_m(Q)/\sqrt{1+\sigma_m(Q)^2} = \sigma_m(D)/\sqrt{1+\sigma_m(D)^2} = p/\sqrt{1+p^2} \in S^0(A^*)/S^{-1}(A^*).$$

On the other hand, at the boundary, we have

$$\sigma_\partial(P) = \sigma_\partial(Q)/\sqrt{1+\sigma_\partial(Q)^2} = p/\sqrt{1+p^2}.$$

Therefore, $\sigma_{full}(P) = p/\sqrt{1+p^2}$ on Ω. Hence the K-theory class $[\widetilde{\sigma}_{full}(P)]$ is obtained from the endomorphism $p/\sqrt{1+p^2}$ defined on TM_0, which obviously extends $p/\sqrt{1+p^2}$ from Ω to the whole of $\overline{A} \supset TM_0$. We obtain that the endomorphism p, and hence also $\sigma_m(D)\widehat{\otimes}1 + 1\widehat{\otimes}V_0$, represents $[\widetilde{\sigma}_{full}(Q)]$.

From the definition of tensor product in K-theory we have that

$$[\sigma_m(D)] \otimes \pi^*[V_0] = [\pi^*(W\widehat{\otimes}E)^+, \pi^*(W\widehat{\otimes}E)^-, \sigma_m(D)\widehat{\otimes}1 \oplus 1\widehat{\otimes}V_0]$$

where

$$\sigma_m(D)\widehat{\otimes}1 \oplus 1\widehat{\otimes}V_0 = \begin{pmatrix} \sigma_m(D^+)\otimes 1 & -1\otimes V_0^- \\ 1\otimes V_0^+ & \sigma_m(D^-)\otimes 1 \end{pmatrix}.$$

It follows that the K-theory class $[\widetilde{\sigma}_{full}(Q)]$, where $Q = D\widehat{\otimes}1 + 1\widehat{\otimes}V_0$, is represented by the same morphism as $[\sigma_m(D)] \otimes \pi^*[V_0]$. So these two classes are equal. □

We shall need the Sobolev spaces $H^m_{\mathcal{W}}(M_0)$ defined by $\mathcal{W}$ (more precisely by the metric determined by $\mathcal{W}$ [1, 2]).

$$H^m_{\mathcal{W}}(M_0) := \{u \in L^2(M),\ Du \in L^2(M_0) \text{ for all } D \in \mathrm{Diff}^m_{\mathcal{W}}(M_0)\}. \tag{35}$$

The space $H^m_{\mathcal{W}}(M_0)$ is the domain of any elliptic pseudodifferential operator in $\Psi^m_{\mathcal{W}}(M_0)$, $m > 0$, acting on $L^2(M_0)$. For $m < 0$ we use duality.

We now show that $\mathrm{ind}(Q^+)$ can be computed using Theorem 2.9.

Theorem 3.7. *Let $Q = D + V_0$ be a Callias-type pseudodifferential operator with a bounded potential V_0 as in Lemmas 3.5 and 3.6. In particular, we assume that V_0 is a smooth potential on M that is invertible on ∂M. Then Q^+ is Fredholm and*

$$\begin{aligned}\mathrm{ind}(Q^+) &= \mathrm{ch}_0[\widetilde{\sigma}_{full}(Q^+)]\pi^* Td(T_{\mathbb{C}}M)[TM_0] = \mathrm{ch}_1[\sigma_{full}(Q^+)]\pi^*_\Omega Td(T_{\mathbb{C}}M)[\Omega] \\ &= \mathrm{ch}_0[\sigma_m(D^+)]\,\mathrm{ch}_0\,\pi^*[V_0]\pi^* Td(T_{\mathbb{C}}M)[TM_0].\end{aligned}$$

Proof. Let $P := Q(1+Q^*Q)^{-1/2}$, as before. Then $P \in \overline{\Psi^0_{\mathcal{W}}(M_0; W \otimes E)}$ is fully elliptic, by the previous lemma (Lemma 3.5). Hence P is Fredholm by Proposition 2.4.

$$\begin{aligned}\operatorname{ind}(Q^+) &= \operatorname{ind}(P^+)\\ &= \operatorname{ch}_0[\widetilde{\sigma}_{full}(P^+)]_0 Td(T_{\mathbb{C}}M_0)[TM_0] = \operatorname{ch}_1[\sigma_{full}(P^+)]\pi^* Td(T_{\mathbb{C}})[\Omega]\\ &= \operatorname{ch}_0[\widetilde{\sigma}_{full}(Q^+)] Td(T_{\mathbb{C}}M_0)[TM_0] = \operatorname{ch}_1[\sigma_{full}(Q^+)] Td(T_{\mathbb{C}})[\Omega],\\ &= \operatorname{ch}_0[\sigma_m(D^+)] \operatorname{ch}_0 \pi^*[V_0]\pi^* Td(T_{\mathbb{C}}M)[TM_0].\end{aligned}$$

by Theorem 2.9 applied to P^+ and Lemma 3.6. □

We are mainly interested in the case when

$$Q = \not{D} + V_0 := \not{D}\widehat{\otimes}1 + 1\widehat{\otimes}V_0, \tag{36}$$

where $\not{D}$ is a Dirac operator acting on the sections of some Clifford bundle W. As before, we assume V_0 is potential (*i.e.,* an odd, self-adjoint, endomorphism of a $\mathbb{Z}_2$-graded bundle E) that is invertible outside a compact subset of M_0 such that V_0 extends smoothly to M, to be invertible at ∂M. In particular $Q \in \Psi^1_{\mathcal{W}}(M_0; W \otimes E)$.

To get an even more explicit formula for the index of coupled Dirac operators $\not{D} + V_0$, let us now that M has a spin^c-structure, with canonical spin^c-bunde S and associated Dirac operator $\not{D}_S$. In particular, M is oriented, and we let $[M] \in H_n(M, \partial M)$ denote its fundamental class. Let $W = S\widehat{\otimes}F$, with F a complex vector bundle over M. Then $\not{D}_F := \not{D}_S \otimes F$ is *the Dirac operator twisted with* F.

Corollary 3.8. *Let $\not{D}_F$ be the Dirac operator twisted with F and $Q = \not{D}_F + V_0$ be the perturbed twisted Dirac operator associated to V_0, where V_0 is a bounded potential invertible at ∂M on an asymptotically commutative spin^c Lie manifold $(M, \mathcal{W})$. Then Q^+ is Fredholm and*

$$\operatorname{ind}(Q^+) = \widehat{A}(M)\operatorname{ch}_0([F \otimes V_0])[M].$$

Proof. It is known classically that

$$p_! \operatorname{ch}_0(\sigma(\not{D}^+_F)) Td(T_{\mathbb{C}}M) = \widehat{A}(M)\operatorname{ch}_0[F],$$

where $p_!$ is integration over the fibre and $\widehat{A}(M) \in H^*(M)$ is the $\widehat{A}$-genus of M (see [35]). The result then follows right away from Theorem 3.7. □

3.4. **The case of unbounded potentials.** In this subsection, we are back to a general (even-dimensional) Lie manifold $(M, \mathcal{V})$. Let

$$T := \not{D} + V := \not{D}\widehat{\otimes}1 + 1\widehat{\otimes}V, \tag{37}$$

where $\not{D} \in \operatorname{Diff}^1_{\mathcal{V}}(M_0; W)$ is a Dirac operator associated to $\mathcal{V}$. We will consider here *unbounded potentials*, in that we assume moreover that, on M_0,

$$V = f^{-1}V_0, \quad f := \Pi x_k^{a_k}, \tag{38}$$

where V_0 is bounded and it extends to a smooth function on M that is invertible on ∂M (at infinity) and x_k are boundary defining functions of the hyperfaces of M with $a_k \in \mathbb{N} = \{1, 2, \ldots\}$. This section contains the hard analysis needed for our main result.

Our first goal is to show that T is essentially self-adjoint with domain a suitable weighted Sobolev space. We want to prove a formula for the index of $T^+ = (\not{D} + V)^+$. Our strategy is to reduce this problem to a question on operators with bounded potential by writing

$$T = f^{-1/2} Q f^{-1/2}, \quad \text{with } Q := f^{1/2} \not{D} f^{1/2} + V_0. \tag{39}$$

In fact, let $\mathcal{W} := f\mathcal{V}$ and let g be the given metric compatible with $\mathcal{V}$. Then $g_0 := f^{-2} g$ is a metric compatible with $\mathcal{W}$ and hence

$$\not{D}_{\mathcal{W}} := f^{1/2} \not{D} f^{1/2}$$

is the Dirac operator associated to the Lie manifold structure defined by $\mathcal{W}$ and metric g_0 [11, 10, 27, 35, 37]. Actually, to identify $\not{D}_{\mathcal{W}}$ with $f^{1/2} \not{D} f^{1/2}$, we need to rescale the volume forms also, a fact that we ignore throughout, in order to simplify the notation.

We then have that $Q \in \mathrm{Diff}^1_{\mathcal{W}}(M; W \otimes E)$ is a Callias-type Dirac operator on $(M, \mathcal{W})$ with a bounded, invertible potential. In particular, it is elliptic and essentially self-adjoint on $\mathcal{C}^\infty_c(M_0)$. As before, we still denote its self-adjoint closure by Q.

We now define *weighted Sobolev spaces* defined by $\mathcal{W}$

$$\mathcal{K}^m_a(M_0) := f^a H^m_{\mathcal{W}}(M_0), \tag{40}$$

where $a \in \mathbb{R}$ and $f = \prod x_k^{a_k}$, as before. If $E \to M$ is a smooth vector bundle, then the spaces $\mathcal{K}^m_a(M_0; E)$ are defined similarly. We remark that all the weighted Sobolev spaces used below are with respect to $\mathcal{W}$. (One can check that $\mathcal{K}^m_a(M_0; E; \mathcal{V}) = \mathcal{K}^m_{a-n/2}(M_0; E; \mathcal{W})$.) We have the following elliptic regularity result from [1].

Theorem 3.9. *Assume that $Q_0 \in \Psi^k_{\mathcal{W}}(M_0; E)$ is elliptic and $h \in \mathcal{K}^s_a(M_0; E)$ is such that $Q_0 h \in \mathcal{K}^{m-k}_a(M_0; E)$. Then $h \in \mathcal{K}^m_a(M_0; E)$.*

Applying this result to $Q = f^{1/2} T f^{1/2} = \not{D}_{\mathcal{W}} + V_0$ we obtain the following.

Lemma 3.10. *Let $h \in \mathcal{K}^s_a(M_0; E)$ be such that $Th \in \mathcal{K}^{m-1}_{a-1}(M_0; E)$. Then $h \in \mathcal{K}^m_a(M_0; E)$.*

We shall also need the following lemma. Before, we remark that it follows from the definitions that multiplication by f^s defines an isomorphism $f^s : \mathcal{K}^m_a(M_0; E_0) \to \mathcal{K}^m_{a+s}(M_0; E_0)$, for any a, s. In particular, if $P \in \Psi^k_{\mathcal{W}}(M_0)$, with $P : H^m_{\mathcal{W}}(M_0; E_0) \to H^{m-k}_{\mathcal{W}}(M_0; E_0)$, then $f^s P f^{-s} : \mathcal{K}^m_s(M_0; E_0) \to \mathcal{K}^{m-k}_s(M_0; E_0)$ is Fredholm if, and only if, P is. Moreover, it is known that $f^s \Psi^k_{\mathcal{W}}(M_0; E_0) f^{-s} = \Psi^k_{\mathcal{W}}(M_0; E_0)$ (Proposition 4.3 [1]), so any such P is also defined as an operator, still denoted by P,

$$P : \mathcal{K}^m_s(M_0; E_0) \to \mathcal{K}^{m-k}_s(M_0; E_0),$$

for any s.

Lemma 3.11. *Let $Q_0 \in \Psi^k_{\mathcal{W}}(M_0; E_0)$, $k \in \mathbb{Z}_+$, be a fully elliptic. Then*

$$Q_{a,b,c} := f^b Q_0 f^c : \mathcal{K}^m_a(M_0; E_0) \to \mathcal{K}^{m-k}_{a+b+c}(M_0; E_0)$$

is Fredholm and its index is independent of m, a, b, and c, in the sense that

$$\operatorname{ind}(Q_{a,b,c}) = \operatorname{ind}(Q_{0,0,0}).$$

Proof. Let us notice first that $Q_{0,0,0}$ is Fredholm due to Proposition 2.8 (since we assumed Q_0 to be fully elliptic). It follows that $f^s Q_{0,0,0} f^{-s} : \mathcal{K}^m_s(M_0; E_0) \to \mathcal{K}^{m-k}_s(M_0; E_0)$ is also Fredholm and $\operatorname{ind}(f^s Q_{0,0,0} f^{-s}) = \operatorname{ind}(Q_{0,0,0})$. Note also that $Q_{a,b,c}$ is indeed well-defined, by the remarks above. Write $Q_a = Q_{a,0,0}$.

Next, we notice that $f^s P f^{-s} - P = f^s(P f^{-s} - f^{-s} P) \in f\Psi^{k-1}_{\mathcal{W}}(M_0; E_0)$ for any $P \in \Psi^k_{\mathcal{W}}(M_0; E_0)$ by the specific form of the Lie algebra of vector fields $\mathcal{W} = f\mathcal{V}$. Moreover $f^s Q_{a-s} f^{-s} - Q_a : \mathcal{K}^m_a(M_0; E_0) \to \mathcal{K}^{m-k}_a(M_0; E_0)$ is compact, since $f\mathcal{K}^m_a(M_0; E_0) \to \mathcal{K}^{m-k}_a(M_0; E_0)$ is compact by [2]. With $s = a$, we conclude that Q_a is Fredholm and also that Q_a and $f^s Q_{a-s} f^{-s}$ have the same index for any s.

If follows that the index of $Q_a : \mathcal{K}^m_a(M_0; E_0) \to \mathcal{K}^{m-k}_a(M_0; E_0)$ is independent of a. Using this with a replaced by $a + c$ and using the fact that $f^s : \mathcal{K}^m_a(M_0; E_0) \to \mathcal{K}^m_{a+s}(M_0; E_0)$ is an isomorphism, we obtain the desired result. □

We shall use this lemma to prove the following crucial result.

Proposition 3.12. *The operators*

$$T \pm iI = \not{D} + V \pm iI : \mathcal{K}^1_1(M_0; W \otimes E) \to \mathcal{K}^0_0(M_0; W \otimes E) = L^2(M_0; W \otimes E)$$

are invertible, and hence T is essentially self-adjoint with domain $\mathcal{K}^1_1(M_0; W \otimes E)$, where all the L^2 and Sobolev spaces are associated to $\mathcal{W}$.

Proof. Let us denote by

$$Q_a := f^{1/2} T f^{1/2} = \not{D}_{\mathcal{W}} + V_0 : \mathcal{K}^{1/2}_a(M_0; W \otimes E) \to \mathcal{K}^{-1/2}_a(M_0; W \otimes E).$$

Then Q_0 is fully elliptic (by Lemma 3.5 and the fact that $\not{D}_{\mathcal{W}}$ is the Dirac operator associated to $\mathcal{W} = f\mathcal{V}$), and hence it is Fredholm. It follows from Lemma 3.11 that Q_a is Fredholm for any a and that its index is independent of a. Since $Q_0^* = Q_0$, we have $\operatorname{ind}(Q_a) = 0$ for all a. Hence

$$\operatorname{ind}(Q_a + \lambda f) = \operatorname{ind}(Q_a) = 0,$$

since multiplication by f is a compact operator $\mathcal{K}^{1/2}_a(M_0; W \otimes E) \to \mathcal{K}^{-1/2}_a(M_0; W \otimes E)$ by [2]. Then

$$T \pm iI = f^{-1/2}(Q_a \pm if) f^{-1/2} : \mathcal{K}^{1/2}_{1/2}(M_0; W \otimes E) \to \mathcal{K}^{-1/2}_{1/2}(M_0; W \otimes E) \tag{41}$$

is also Fredholm of index zero.

We recall that $\mathcal{K}_a^m(M_0; W \otimes E)$ is the dual of $\mathcal{K}_{-a}^{-m}(M_0; W \otimes E)$, with the duality pairing being obtained from the L^2-inner product by continuous extension. Then the "L^2-estimate"

$$((T \pm i)u, u) = (Tu, u) \pm i(u, u)$$

and $(Tu, u) \in \mathbb{R}$ (since T is symmetric between the indicated spaces in Equation (41)) show that $T \pm iI$ are injective for $a = 0$. Since they have index zero, they induce isomorphisms

$$T \pm iI : \mathcal{K}_{a+1/2}^{m+1/2}(M_0; W \otimes E) \to \mathcal{K}_{a-1/2}^{m-1/2}(M_0; W \otimes E) \tag{42}$$

for $a = 0$ and $m = 0$.

Now for an arbitrary a, the induced operator will still have index zero (by Lemma 3.11). Since for $a \geq 0$ it will still be injective, it follows that it will be an isomorphism for all $a \geq 0$. Since for $a < 0$ the resulting map is dual to the one for $-a$, we obtain that $T \pm iI$ of Equation (42) are isomorphisms for all a and $m = 0$. We can extend this isomorphism to any $m \geq$ by elliptic regularity (Lemma 3.10) and this completes the proof by taking $a = m = 1/2$. □

We shall extend T to a self-adjoint operator denoted by the same letter. We are ready now to compute the index of

$$T^+ = (\not{D} + V)^+ : \mathcal{K}_1^1(M_0; (W \widehat{\otimes} E)^+) \to \mathcal{K}_0^0(M_0; (W \widehat{\otimes} E)^-),$$

where $\not{D}$ is the Dirac operator on the (arbitrary) even dimensional Lie manifold $(M_0, \mathcal{V})$ and $V = f^{-1}V_0$ is an unbounded potential as in (38). Let $\pi : \overline{TM} \to M$ and $\pi_\Omega : \Omega = \partial \overline{A_\mathcal{V}} \to M$ be the natural projections and $Td(T_\mathbb{C}M)$ be the Todd class of the complexified tangent bundle of M.

Theorem 3.13. *The operator $T^+ = (\not{D} + V)^+$ is Fredholm and its index is given by*

$$\begin{aligned} \mathrm{ind}(T^+) &= \mathrm{ch}_0[\widetilde{\sigma}_{full}(T^+)]\pi^* Td(T_\mathbb{C}M)[TM_0] = \mathrm{ch}_1[\sigma_{full}(T^+)]\pi_\Omega^* Td(T_\mathbb{C}M)[\Omega] \\ &= \mathrm{ch}_0[\sigma_1(\not{D}^+)]\,\mathrm{ch}_0\,\pi^*[V_0]\pi^* Td(T_\mathbb{C}M)[TM_0]. \end{aligned}$$

Proof. Let $Q_1 = T^+ f$, where $f = \prod x_k^{a_k}$ as above and is regarded as a multiplication operator. Then $Q_1 = f^{-1/2} Q^+ f^{1/2}$, where

$$Q^+ := f^{1/2} T^+ f^{1/2} = (\not{D}_\mathcal{W} + V_0)^+ : \mathcal{K}_0^1(M_0; (W \widehat{\otimes} E)^+) \to \mathcal{K}_0^0(M_0; (W \widehat{\otimes} E)^-)$$

is fully elliptic (by Theorem 3.7 and the fact that $\not{D}_\mathcal{W}$ is the Dirac operator associated to $\mathcal{W} := f\mathcal{V}$). Then the operators Q_1 and Q^+ have the same index, by Lemma 3.11. By ellipticity,

$$(1 + Q_1^* Q_1)^{1/2} : \mathcal{K}_0^1(M_0; (W \widehat{\otimes} E)^+) \to L^2(M_0; (W \widehat{\otimes} E)^+)$$

is an isomorphism since the domain of any elliptic operator $P \in \Psi^m_{\mathcal{W}}(M;E)$ is $H^m_{\mathcal{W}}(M_0, E)$. Therefore $f(1+Q_1^*Q_1)^{-1/2} : L^2(M_0; (W\widehat{\otimes}E)^+) \to \mathcal{K}^1_1(M_0; (W\widehat{\otimes}E)^+)$ is an isomorphism as well. Proposition 3.12 then yields that $T^+ : \mathcal{K}^1_1(M_0) \to L^2(M_0)$ has the same index as

$$T^+ f(1+Q_1^*Q_1)^{-1/2} = Q_1(1+Q_1^*Q_1)^{-1/2} : L^2(M_0; (W\widehat{\otimes}E)^+) \to L^2(M_0; (W\widehat{\otimes}E)^-),$$

and, in particular, they are both Fredholm.

We have thus obtained that the operator $T^+ = (\not{D} + V)^+$ is Fredholm and has the same index as $Q^+ := f^{1/2} T^+ f^{1/2}$. Moreover, the principal symbols of T^+ and Q^+ define the same K-theory classes, by homotopy invariance, as do the symbols of $\not{D}$ and $\not{D}_{\mathcal{W}}$, and hence

$$\begin{aligned}\operatorname{ind}(T^+) = \operatorname{ind}(Q^+) &= \mathrm{ch}_0[\sigma(Q^+)]\pi^* Td(T_{\mathbb{C}}M)[TM_0] = \mathrm{ch}_1[\sigma(Q^+)]\pi^*_{\Omega} Td(T_{\mathbb{C}}M)[\Omega] \\ &= \mathrm{ch}_0[\sigma(T^+)]\pi^* Td(T_{\mathbb{C}}M)[TM_0] = \mathrm{ch}_1[\sigma(T^+)]\pi^*_{\Omega} Td(T_{\mathbb{C}}M)[\Omega] \\ &= \mathrm{ch}_0[\sigma(\not{D}^+)]\,\mathrm{ch}_0\,\pi^*[V_0]\pi^* Td(T_{\mathbb{C}}M)[TM_0],\end{aligned}$$

by Theorem 3.7 applied to Q^+ and homotopy invariance. □

We also obtain the following more explicit calculation similar to Corollary 3.8.

Corollary 3.14. *Let $\not{D}_F$ be the Dirac operator twisted with F and $T = \not{D}_F + V$ be the perturbed twisted Dirac operator associated to $V = f^{-1}V_0$, where V_0 is a bounded potential on M invertible at ∂M for a spin^c Lie manifold $(M, \mathcal{V})$. Then T^+ is Fredholm and, using the notation of Corollary 3.14*

$$\operatorname{ind}(T^+) = \widehat{A}(M)\,\mathrm{ch}_0([F \otimes V_0])[M] = \widehat{A}(M)\,\mathrm{ch}_0([F \otimes V])[M].$$

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Departamento de Matemática, Instituto Superior Técnico, UTL, Lisbon, Portugal
E-mail address: ccarv@math.ist.utl.pt

Pennsylvania State University, Math. Dept., University Park, PA 16802, USA, and Inst. Math. Romanian Acad. PO BOX 1-764, 014700 Bucharest Romania
E-mail address: nistor@math.psu.edu